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UNIVERSITY OF OKLAHOMA BULLETIN

THE TEACHING OF HIGH SCHOOL MATHEMATICS

BY
JASPER O. HASSLER
PROFESSOR OF MATHEMATICS
UNIVERSITY OF OKLAHOMA

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THE UNIVERSITY OF OKLAHOMA
Norman, Oklahoma

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By
JASPER O. HASSLER

FOREWORD

This bulletin has grown out of the lectures given by me at the University of Oklahoma in the course on the teaching of high school mathematics. It is published by the University with the hope of helping the high school teachers of mathematics in the state.

The purpose of Part One is to give an insight into the true nature of the subject we call mathematics, and a brief view of the historical development of the most elementary parts of it. The reason for including this in a text on teaching is explained by the following extract from §7 of the bulletin:

> We would not think much of a teacher of a class reading in the Third Reader who had never read beyond the Fourth Reader herself and knew nothing of literature.
>
> It is true, nevertheless, that there is a misconception that if anyone has had mathematics beyond what he teaches he is a satisfactory teacher. Until this diseased state of mind is healed we will continue to have indifferent teaching of mathematics. Every teacher of mathematics who expects to continue as such should feel obligated to make further (and then further) acquaintance with his subject.

Part Two deals with standard, time-tested methods of teaching mathematics and recent movements toward the improvement of methods of teaching. Much space is given to recent tendencies toward the reorganization of the subject matter of high school mathematics. The suggestions on the teaching of algebra and geometry are the outgrowth of ten years of experience in teaching those subjects to boys and girls in the high school.

It will be better (though not absolutely necessary) to read entirely through Part One before reading Part Two even if the mathematical preparation of the teacher is so limited that there are parts which are new and hard to understand. There are some new things that can only be learned by reading material which is neither easy nor familiar. No teacher of mathematics worthy of the name should be afraid to tackle any of the subject-matter contained in this bulletin. Many references are made in Part Two to material in Part One.

Acknowledgment is due to my colleagues, Professor Edgar Meacham, who read the manuscript of Chapters I, III, and V and made many valuable suggestions as to content, Miss Dora McFarland, who read all the proof and made many helpful suggestions as to style, Dr. J. W. Scroggs and Dean S. W. Reaves for their friendly cooperation and kindly suggestions, and especially to the staff of the University Print Shop, without whose ingenuity and kindly efforts with a limited variety of equipment this bulletin could never have been published with such accuracy and attractiveness.

We apologize for the lack of italic type of the standard size used in this bulletin. Black-face type and quotation marks have been sometimes used instead for emphasis. The literal parts of equations have also been printed in roman type.

It is the desire of the Extension Department that every teacher of mathematics in the state should have a copy of this bulletin. Any teacher who has not received one should write for it.

J. O. HASSLER.

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THE TEACHING
OF
HIGH SCHOOL MATHEMATICS

PART ONE
PRIMARY CONCEPTS OF MATHEMATICS

CHAPTER I.

THE ORIGIN AND NATURE OF MATHEMATICS

1. **Whence Came Mathematics?** Long before the geometers of ancient Greece discovered and proved with mathematical rigor that regular hexagons may be made to fill a plane space without overlapping or leaving space between them, the economical bees were making hexagonal honey cells. For thousands of years before it was announced as an "axiom" that "a straight line is the shortest distance from one point to another", human beings and animals were "cutting corners" in order to shorten distances. Before the advent of human life the forces of nature were building up sedimentary rocks in parallel strata. Before **any** life existed on this planet, the bodies of the solar system whirled on in their endless journeys about the sun in ellipses—geometric curves. Even before the planets were formed every two particles of planetary matter were attracting each other by a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them, a relation simply expressed by an algebraic formula. The contact of man with the forces of nature, his experiences and observation of events happening daily about him, inspired him to formulate the observed laws governing the events and classify the facts and phenomena in a scientific manner. Thus came mathematics. We may say that it grew out of a need for a mode of expression for the findings of quantitative thought about the natural phenomena of space. It has been developed into a logical science.

2. What is Mathematics? Let us first consider what it is not. A teacher of college mathematics had just returned to his native town after three years' study in Germany which had resulted in his receiving the degree, Doctor of Philosophy, at the University of Heidelberg. His brother was accosted on Main Street by a friend of the family who had heard of the return of Robert, the student and scholar.

"They tell me 'Bob' has just got back from Germany", said the kindly friend.

"Yes, he returned yesterday", replied the brother.

"What has he been doing over there all these years?"

"Studying mathematics."

"Nothing but mathematics?"

"That's all."

The man gazed meditatively across the street at the blank wall of a brick building. He tried to grasp the immensity of the thing. Finally he blurted out, "I'll bet he could tell in a minute how many bricks there are in that wall!"

It must have been experiences with people who had misconceptions of the nature of mathematics similar to that related above that caused the late Professor Maschke of the University of Chicago to say, "Mathematics is the science of how not to compute." Of course Professor Maschke did not mean to give a scientific definition of mathematics in that statement. Probably the following examples will illustrate what he meant.

A mechanic wishes to find the weight of a cylindrical pipe whose inside and outside diameters are fifteen and seventeen inches, respectively. Knowing the weight per cubic inch he must compute the volume and must multiply the area of a cross section by the length to get the volume. In all probability he would compute the area of the outer and inner circle separately and find the difference thus:

$$\begin{aligned} 3.14(8.5)^2 - 3.14(7.5)^2 &= (3.14)(72.25) - (3.14)(56.25) \\ &= 226.9 - 176.6 = 50.3 \\ &\quad \text{(approximately)} \end{aligned}$$

An alert student of mathematics will see the possibility of using factoring and write, or think:

$3.14(8.5 + 7.5)(8.5 - 7.5) = (3.14)(16) = 50.2$ (approximately)

It can be done without pencil and paper.

A boy once wanted to find the sum of all the integers from 1 to 150 and promptly wrote them down and added them. Later he learned how to use the formula for the sum of an arithmetic progression $s = \frac{n}{2}(a + 1)$, by means of which the sum is found to be the product, 75 × 151.

A graduate student in a class in mechanics was once computing the resultant of a force and obtained the expression, $\sqrt{(500 \cos 60^\circ)^2 + (500 \cos 30^\circ)^2}$, which he reduced as follows: $\sqrt{(250 \cdot 1)^2 + (250 \cdot \sqrt{3})^2} = \sqrt{62500 + 187500} = \sqrt{250000} = 500.$ Had he seen the common factor $(500)^2$ in the radicand of the first expression and been alert enough to realize that by changing $\cos 60^\circ$ to $\sin 30^\circ$ it could be written

$500 \sqrt{\sin^2 30^\circ + \cos^2 30^\circ} = 500$, (since $\sin^2 30^\circ + \cos^2 30^\circ = 1$) he would have avoided much computation.

A professor of Education in a university had a list of about 7000 words (used in a certain textbook under investigation) which he arranged in order of the frequency of their occurence in the book. From this list he wished to select words according to the following plan: choose the 1st, 2nd, 4th, 7th, 11th, 16th, etc., the intervals between the words increasing by one with each choice. How could he know how many words there would be in his chosen list without counting through the entire list of words? Let n be the number of words chosen and N the number in the original list of the last word chosen. An examination of the statement of the method of choice shows that N is one greater than the sum of the integers from one to n. Since there are 7000 words, the equation,

$$1 + \frac{n}{2}(1 + n) = 7000,$$

will have as its positive root one less than the number of words chosen.

The preferred method of solving each of the four problems just mentioned involves computation but it avoids more tedious computation. The ability to sense the situation in each case and to use the better method may be regarded as ability to use mathematical knowledge, or, shall we say, it is mathematical ability.

Mathematics may involve computation but it is not merely the art of computation. It is not merely a compendium of rules for solving problems. We will consider further some of the characteristics of a mathematical science and later study some definitions of mathematics.

3. Two Important Characteristics of a Mathematical Science.

A pure mathematical science is ABSTRACT.

A mathematical science can exist in the abstract without any applications. Much of mathematics has been developed because of the need for it in concrete, everyday problems. Much more has been developed for the love of creative research and as yet applies to nothing that touches one's ordinary life. Applications have been found for some abstract mathematical theory years after its development; but the mathematics exists apart from its applications. The following quotation from Poincare[1] is to the point.

> Les Mathematiciens n'etudient pas les objets, mais des relations entre les objets; il leur est donc indifferent de remplacer ces objets par d'autres, porvu que les relations ne changent pas. Le mattiere ne leur importe pas, la forme seule les interesse. (Mathematicians do not study objects, but the relations between objects; hence it is immaterial to them if these objects are replaced by others, provided that the relations do not change. Matter is of no consequence to them; form alone interests them.)

A pure mathematical science is ARBITRARY.

The conclusions of a pure mathematical science do not exist until some mind produces them indirectly by the assumption of certain postulates and hypotheses from which the conclusions necessarily follow or by direct expression of the facts themselves. Take the simple, well-known relation $ab = ba$ (called the commutative law of multiplication) that we use without question in arithmetic, have laid down for us in general terms in algebra, and demonstrated (?) in some elementary algebras. It is true only in the sense that we agree that the kind of numbers which we propose to use in elementary algebra are such numbers that two of them may have their order changed in multiplication without changing their product. There are some numbers used in advanced algebras for which this commutative law is not true. For an illustration of this see §38.

4. The Foundations of a Mathematical Science. A mathematical science is built up from certain undefined terms and unproved propositions. It is clear that we can not define the first term in our science. We would have no concepts in terms of which we might express our definition. We must have then some undefined terms. They represent notions which are clearer to the human mind than any verbal description can make them. Inasmuch as a deductive proof must have known truths as a foundation, there must be a starting point of unproved propositions. Sometimes these have been called axioms. Mathematically speaking, an axiom is a statement that we do not prove, but for the purpose of the science its truth is assumed. As an illustration let us consider the development of a miniature plane geometry consisting of one theorem.

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[1] H. Poincare, *La Science et L'Hypothese,* p. 32, Flammarion, Paris, 1908.

The undefined notions are **point,**[2] **straight line, plane, direction,** and the **rotation** in a plane of a straight line about its point of intersection with another straight line. We will make the following definitions:

A **plane angle,** formed by two straight lines, called **sides,** is the amount of rotation in a plane in a given direction about their common point necessary to make one of them coincide with the other. The common point is called the **vertex** of the angle.

A **straight angle** is an angle whose sides extend in opposite directions from the vertex, forming a straight line.

The assumptions, or unproved propositions, shall be:

1. Quantities equal to the same or equal quantities are equal.
2. If equals are subtracted from equals the remainders are equal.
3. A whole is equal to the sum of all its parts, provided it is divided into positive parts only and no parts overlap.

The first theorem shall be:

If two straight lines intersect, the opposite angles formed are equal.

The proof of this theorem by the usual method, which it is not necessary to copy here as the reader can easily supply it, involves the use of exactly all the definitions and assumptions given above. If other theorems are desired the list of definitions and assumptions must be extended.

It must be noted that the viewpoint from which the science of mathematics is here represented is not the only one. Conceptions of the same thing by different people differ according to their intellectual abilities. The pupil in secondary mathematics may not be expected to get a mature conception of mathematics as a logical science, but the teacher should certainly have it in order that the pupil's modified conception may not be misleading.

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[2] We may say that a point is that which has position, without length, breadth, or thickness but we are not defining it. We are simply making an explanation that is supposed to help clarify the idea in the pupil's mind. The notion of point can not be conveyed through mathematical terms so as to make it any more definite than it naturally is.

5. Outline of Mathematics. We will now consider a classification of the more familiar subjects in the science of mathematics. This is not an exhaustive classification. Under applied mathematics are named just enough subjects to illustrate fields of application of all three of the principal divisions of pure mathematics.

A. Pure mathematics
    1. Algebra
        Arithmetic, number theory, group theory, invariants, etc.
    2. Geometry
        From four viewpoints it is: Euclidean or non-Euclidean; metric or projective; synthetic or analytic; integral or differential.
    3. Analysis
        Calculus, differential equations, function theory, etc.
B. Applied mathematics
        Insurance and investment mathematics, astronomy, mechanics, etc.

The four viewpoints in geometry are not mutually exclusive. The geometry we study in high school is Euclidean, metric, synthetic, and integral. The geometry of conic sections as we usually study it in the first or second year in college is Euclidean, analytic, metric, and integral. The terms synthetic and analytic refer to the method of developing the subject. We have an example of what is meant by differential geometry when by means of differential calculus we study such properties of curves as slope, maxima and minima, nature of curvature, and order of contact by means of derivatives. Non-Euclidean geometry is treated in Chapter V.

6. Some Definitions of Mathematics.

"Mathematics is the science which draws necessary conclusions."—Benjamin Peirce[3].

Professor Peirce meant by the word necessary that the conclusions of mathematics, after the postulates and hypotheses are fixed, must follow. From a given group of axioms there arise certain theorems and no others. They are there whether we find them or not. If, for example, we adopt the axioms and postulates of Euclid, the Pythagorean Theorem (the square on the hypotenuse is equal to the sum of the squares on the other two sides) can be demonstrated, and it was a truth as much during the centuries before its discovery as it is now. The development of the miniature geometry in §4 is also an illustration of this. Furthermore, for a given set of hypotheses in a theorem the conclusion reached is the only conclusion. No different conclusion can be reached unless some changes are made in the hypotheses.

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[3] *American Journal of Mathematics,* Vol. 4 (1881), p. 97.

"A mathematical science is any body of propositions arranged according to a sequence of logical deductions."[4]

"Mathematics is the class of all propositions of the form, P implies Q."[5]

These two definitions also emphasize the logical side of mathematics. The latter tersely states that mathematics is made up of syllogisms of deductive logic. For example, equal sides in a triangle imply equal angles in the triangle, or, the fact that a rational integral equation has a complex root implies that there is another complex root of the equation.

"Mathematics is the science in which we never know what we are talking about nor whether what we say is true."—Bertrand Russell.[6]

This definition is not so facetious as one thinks when he first reads it. The symbols of mathematics are abstract and arbitrary and the same symbol may represent different values in different problems. We have symbols which we say stand for the "unknowns" in a problem. Hence in a way we do not know all we are talking about. Inasmuch as the proofs of our propositions depend, according to "a sequence of logical deductions", entirely on the truth of our axioms or postulates, "what we say is true" only if the axioms are true and their truth is arbitrarily assumed in the beginning (§4).[7]

We shall conclude this list of definitions of mathematics with the one given in the Oxford Dictionary; namely, "The abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations."

It will be noticed that this is the only definition given that refers to numbers. As may be supposed by the nature of the definitions given, arithmetical operations form a very small (even if very important) part of the science of mathematics; and even the principles and processes of arithmetic are deduced in a logical manner from fundamental assumptions. This is discussed in Chapter III.

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[4] J. W. Young, *Fundamental Concepts of Algebra and Geometry,* p. 2.

[5] Bertrand Russell, *The Principles of Mathematics,* p. 1.

[6] *International Monthly,* Vol. 4 (1901), p. 84.

[7] If any reader has never considered the axioms of geometry other than as "indisputable truths," let it be said that Euclid's axioms are merely statements of relations that seem to be undeniably true only because our experiences find no contradictions. The word "axiom" is derived from a Greek word meaning "that which is thought worthy." This subject is treated more at length in Chapter V.

7. **What This Means to a Teacher of Secondary Mathematics.** Suppose a party of tourists were setting out for a long trip into a comparatively unknown and unsettled region. Suppose no guide is at the time available for more than the first day's journey. Suppose that of two guides under consideration one has never been further than one day's journey on this particular tour while the other has been over the whole route. It goes without saying that the tourists would prefer the services of the guide who knows all about the trip ahead of them beyond the point to which he can take them. They need to be guided in their preparations for the journey by one who knows just what their future needs will be. It seems to me that if a class of pupils is to enter through the doors of algebra and geometry into the domain of the oldest, most used, and most perfectly developed science known to the human race, they should have for a guide one who knows what is inside, far beyond the point to which he expects to lead his class.

We would not think much of a teacher of a class reading in the Third Reader who had never read beyond the Fourth Reader herself and knew nothing of literature.

It is true, nevertheless, that there is a misconception that if anyone has had mathematics beyond what he teaches he is a satisfactory teacher. Until this diseased state of mind is healed we will continue to have indifferent teaching of mathematics. Every teacher of mathematics who expects to continue as such should feel obligated to make further (and then further) acquaintance with his subject.

It is frequently true that as we study mathematics in college, subject by subject, we do not get a broad, comprehensive view of the whole field nor do we grasp the full significance of the term "mathematics". It is the purpose of Chapters III and V to give in a very simple manner a few primary concepts of parts of mathematics, with references for further study for those who will devote time to it. The subjects are those most closely related to the elementary field. Any teacher considering these topics thoughtfully will awake to a better outlook on mathematics. References to these chapters will be made frequently in Part Two of this bulletin.

References.[8]

J. W. Young, Fundamental Concepts of Algebra and Geometry, Lectures I and IV.

Miller, Historical Introduction to Mathematical Literature, 78-86.

J. W. A. Young, The Teaching of Mathematics in the Elementary and Secondary Schools, 22-31.

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[8] A complete list of all references, with publishers, is given at the end of this bulletin.

CHAPTER II.

HISTORY OF THE DEVELOPMENT OF ARITHMETIC AND ALGEBRA

8. **Introduction.** The purpose here is not to present a historical treatise as a part of this bulletin. The references given at the close of this chapter are recommended to any reader who desires a complete presentation of the subject of the history of mathematics. Before studying the methods of teaching any subject in mathematics the teacher should have a thorough knowledge of the subject and its relation to advanced mathematics as well as to practical applications, a clear conception of the mathematical foundations underlying the subject, and a vision of its historical development. The only topics presented here are those which are considered by the student of high school mathematics. It is intended to furnish the teacher of these subjects with information concerning the origin and development of the familiar terms, processes, and symbols of elementary mathematics. The study of the evolution of mathematical knowledge shows in a very convincing manner its vital connection with the progress of human civilization. The student of high school mathematics is able to see this and it is only fair that he should have the opportunity.

9. **Primitive Methods of Counting.** Seldom does it occur to us, as we operate so readily with our system of numbers, that primitive man had difficulty in counting and writing numbers. His first crude attempt to convey by means of written symbols the idea of "two men" was doubtless by drawing two symbols that looked like men. If delivering the message orally he would hold up two fingers to represent two men. He soon learned that two straight marks could be used to represent two men, two horses, two days, two journeys, etc. Extending the process, he soon learned that he could use as a symbol for any number of things an equal number of straight marks. He had learned to express the idea of a one-to-one correspondence between the things of two groups, which is at the basis of the meaning of number[9]. The representation of large numbers by such a scheme

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[9] See §25.

was cumbersome and out of the need for a briefer scheme were born our number symbols.

Devising names for numbers has been as difficult as devising the symbols, if not more difficult. Some aborigines of Australia have not developed number names beyond two. Their entire number vocabulary is contained in words equivalent to "one", "two", "two-one" (3), "two-two" (4), "many", and "very many". Many other tribes in Australia, Africa, and South America have no distinct names beyond four or five.

The convenience of the fingers and toes forced them into use in counting and many are the evidences of man's use of his hands and feet as the first calculating machine. In both Greek and Latin the words for hand have number meanings attached to them. We are all familiar with the double meaning of our word "digit". In the Russian language the word for five is "piat" and the word for fist is "piast". The Kiowa Indians and kindred tribes in Oklahoma have long used as a symbol for five the figure of a closed hand. The Babylonian symbol for ten is thought by some to have represented two hands held in prayer with the fingers pressed together.

The Dakotas counted by tens (fingers of both hands) and bent down one finger when ten was counted. Three fingers bent down would represent thirty. When all the fingers had been bent down a circle was drawn to indicate that they had gone "around the circle" of the fingers. The circle was their accepted symbol for 100. In many cases it was also used as the symbol for any number greater than 100, which to them was beyond power of expression. It was "many". The mathematician of to-day has a symbol whose meaning is closely akin to the "many" of primitive man. It is ∞ (infinity).

Some Indian tribes in South America used the ten toes in counting. Their word for twenty was identical with the expression for "one Indian". The phrase, "One to the hands of the other Indian" meant twenty-one, that is, as many as one Indian had fingers and toes and one finger of another Indian. With some tribes "half a man" was the expression for ten. The natives of Greenland say, "a man ended" for twenty. For higher numbers they count more men. For example, fifty-three would be expressed as "on the third man on the first foot three".

There is no doubt that our decimal system owes its existence to the number of our fingers and it is equally certain, as has often been expressed, that if we had been created with six fingers on each hand we would have had a duodecimal system to-day. Tylor[10] says, "The case is the not uncommon one of higher civilization bearing evident traces of the rudeness of its origin in ancient barbaric life".

10. Systems of Notation. Besides the decimal system there are found examples of quinary and vigesimal systems among various uncivilized tribes. Conant[11] says, "With the exception of a small number of isolated cases * * * * * * it may be laid down as a universal law that every language containing a number system extending beyond five reveals the use of one of the three numbers, five, ten, or twenty as the base of that system".

Conant names some tribes of Africa and Northern Siberia that have preserved pure quinary systems. The Greeks of Homer's time had a quinary system auxiliary to the decimal system[12]. Homer uses the expression "to five", meaning "to count". The familiar Roman system with its I, V, VI, X, XI, L, LX, etc., presents a typical combination of quinary and decimal systems.

The influence of twenty as a base in counting among our Anglo-Saxon ancestors is shown by the use of the word "score" in such expressions as "three score and ten", as well as by our present method of counting to twenty before adopting a purely decimal system of notation. The modern French "quatre-vingt" (four twenties) for eighty shows the influence of an earlier vigesimal notation. History shows that the quinary and vigesimal systems give way to the decimal in the advance of civilization.

The question next arises as to whether or not the decimal will eventually yield to the duodecimal, which many insist would be a much more convenient system. With 12 as a base we would call 13 by some name that would mean $12 + 1$ instead of $10 + 3$, as the name does now, and write it 11, the left figure representing one "twelve" instead of one "ten". Two twelves would be written as 20 and the square of twelve (twelve twelves), 100. The numbers 2, 3, 4, and 6 would all be exact divisors of the base and the frequently used halves, thirds, fourths, and sixths would be expressed duodecimally as .6, .4, .3, and .2. The only unit fractions with denominators less than ten which would produce circulating duodecimals would be 1/5 and 1/7 as contrasted with the fractions 1/3, 1/6, 1/7, and 1/9 which in our present system produce circulating decimals. The difficulty attending the introduction of the metric system into common use presages the impossibility of our ever changing to a duodecimal system of notation. What a pity we were not created with six fingers on each hand!

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[10] *Primitive Culture,* I, 272.

[11] *Smithsonian Report,* (1892) p. 591.

[12] Heath, *History of Greek Mathematics,* I, 26.

A few isolated instances have been reported where such numbers as four, six, eight, nine, and twelve have been used as bases, or at least as auxiliary bases. A very remarkable case is that of the Maoris of New Zealand whose number base is eleven!

The number system of the ancient Babylonians had 60 for its base. It has been suggested that this originated in their division of the circle into 360 parts (the number of days, according to their reckoning, required for the sun to make its apparent circular journey eastward through the stars—the course of the year). The regular inscribed hexagon, with sides equal to the radii, made of the circle six natural divisions of 60 degrees each. The next step was to divide each degree into 60 minutes and each minute into 60 seconds. This so influenced their dealing with numbers that they came to use in all cases the sexagesimal system.

11. The Abacus. We have mentioned that man made of his hands the first calculating machine. The next oldest calculating machine is probably the abacus.

Its origin is not exactly known but it probably existed before the time of Plato. Forms of the abacus are still used by the Chinese, Japanese, Russians, and other peoples of Eastern Europe and Asia.

After the idea of the decimal system took root and man began to deal with the larger numbers it was necessary to have some method of recording the number of tens counted, or even the number of "ten-tens". We do it easily now by our place system wherein we make the second figure from the right stand for tens, the next for hundreds, etc. In adding we carry the number of tens to the next column. But our system was possible only after the zero had been included among the digits, which was certainly not earlier that the ninth century. Before the perfection of our (Hindu-Arabic) system of writing numbers the abacus was invented as an aid in recording large numbers. The simplest form of the abacus is a frame with parallel rods on which slide counters, those on the first rod representing units, the second tens, etc. If on such an abacus we slide forward three counters on the first rod, none on the second, five on the third, and two on the fourth, the number represented is 2503.

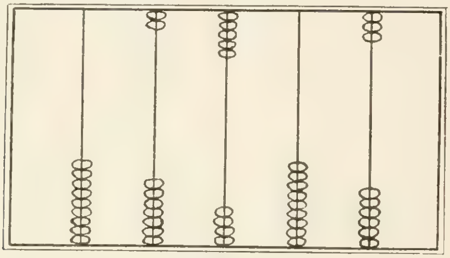

FIGURE 1

One form of the abacus used by the Romans had extra rods on which the counters represented fractions.

Some forms of the Roman abaci were designed so as to use a quinary system as well as a decimal system. Each rod had two sections the lower of which had five counters and the upper, one or two. If one counter of the upper section was used and two of the lower the number represented would be seven. The **suanpan** of the Chinese and the **soroban** of the Japanese were also designed so as to be used with a quinary system.[13]

12. **Primitive Number Symbols.** Reference has been made (§9) to the use by the American Indians of a figure of a hand for five and to the Babylonian symbol for ten representing (according to some) two hands.

Although primitive man has in numerous instances used parts of the body in connection with the spoken name for numbers, the written symbols seem to have been, in general, arbitrarily devised.

The Egyptian hieroglyphic symbol for one was a vertical line; for ten a symbol like an inverted capital U; for one hundred a spiral shaped figure not unlike a capital C reversed. In Fig. 2 is shown the number 2, 143, 257 as written in Egyptian symbols.[14]

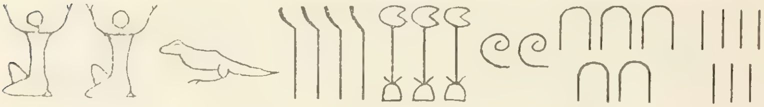

FIGURE 2

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[13] Smith-Mikami, *A History of Japanese Mathematics.*

[14] Peet, *The Rhind Mathematical Papyrus,* p. 11.

The Egyptians used the additive principle in writing numbers but wrote the characters to read from right to left. If Fig. 2 is held before a mirror a truer representation of the number will be seen as written in Egyptian symbols. From this the reader can discern the symbols for 1000, 10,000, 100,000, and 1,000,000. The symbol for 1,000,000 is supposed to represent a man in astonishment.

The Greeks earlier than the third century before Christ used the initial letters of the words for five, ten, and one hundred as the symbols for these numbers; Π for 5, Δ for 10, and H for 100, retaining the short vertical bar for one. They used the additive principle in combining symbols, writing HΔΔΠIII for 128. A small Δ inside of Π stood for 50; a small H inside of Π for 500. This quinary modification of the decimal system enabled them to save much writing.

We are all familiar with the Roman notation, the plan of which is very similar to that of the early Greeks.

Many of the North American Indian tribes used a system of numbers similar to the Roman, but it was limited to two main symbols. They wrote a short vertical bar for one and signs like a plus sign for ten. For 32 they would write ++\+||.

The Mayas of Central America, whose civilization is believed by some to have reached back beyond 3000 B. C., had a place system of writing numbers with simple symbols. The symbol for one was a dot (·) for five a bar (————), and

$$\cdot\ \cdot\ \cdot\,,\quad \overset{\cdot\ \cdot}{\text{————}}\,,\ \text{and}\ \overset{\cdot\ \cdot\ \cdot}{\overset{\text{————}}{\text{————}}}$$

stood for 3, 7, and 13, respectively. They had a crude symbol for zero and wrote the higher numbers according to a vigesimal system.

13. The Origin of Our Present Number Symbols and Names. There has been a great deal of speculation concerning the origin of the symbols for the digits. Various writers have shown how these symbols have evolved from this or that set of forms. For example, some propose that 8 is derived from a method of laying eight sticks in two squares of four each, one above the other. Apparently some have been too eager to confirm their theories to adhere to authentic records. There seems to be no doubt that our present digit symbols originated in India. From the Hindus the Arabs learned them and, in turn, passed them over to Europe. During the centuries of their use by the Indians and Arabs they went through various stages of evolution. It is possible that the symbol 2 may have grown out of two short parallel bars through a form like a z, and that 3 grew out of three short parallel bars. There seems to be no definite connection between the forms of the other digit symbols (with the possible exception of zero — of course the origin of 1 is obvious) and earlier forms suggesting the numbers of things each represents. It is quite possible that the first forms were devised arbitrarily by some scholar, or group of scholars, of ancient India. It may be that they were derived from the initial letters of the words for the numbers.

The symbol 0 (zero), the latest of the ten to be introduced, certainly does not come from obscure antiquity. It is first found in a Hindu inscription dated 876 A. D. In the inscriptions of earlier date a dot was used to indicate the absence of value. The introduction of this dot, followed later by 0, enabled the Hindus to complete the development of the place system of writing numbers which was later passed on to Europe and which almost eliminated the abacus. Whether the 0 "grew" from the dot or was adopted from a round symbol that had been used for ten is not certain. It is true that the Greeks used the symbol *o* (omicron) to indicate the absence of number, but it was not admitted into the family circle with the other numbers.[15]

In primitive languages, which abound richly in metaphor, such words as moon or earth have been used to represent one, because there is but one moon or one earth. Words like eye, wing, or arm, for a similar reason have been used for two. The Hindus used for four the word for oceans (they recognized four of them) and for five the word for "senses". Concerning the genesis of our own words for the first nine numerals nothing is known for certain. The Hindu name for the zero symbol meant "void". The Arabic name, possibly derived from this Hindu name, was the verbal parent of our word cipher.

14. Fractions. The ancient Egyptians apparently could not conceive of fractional parts beyond the unit part. Fractions like 2/3, 5/8, etc., did not appear in Egyptian mathematics, all their fractions having one for the numerator. In writing them they wrote the denominator with a dot above it to indicate that it was to be considered as a fraction. For 2/3 they would write (not using our symbols of course) $\dot{2} + \dot{6}$, meaning 1/2 + 1/6; for 5/8, $\dot{2} + \dot{8}$, meaning 1/2 + 1/8; etc. The use of the dot as the unit-numerator symbol is also found in the eighteenth century.

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[15] Smith-Karpinski, *The Hindu-Arabic Numerals.*

The Babylonians extended the use of the sexagesimal base of their number system to fractions, using only powers and multiples of 60 as denominators. They expressed fractions without denominators as we do decimal fractions. In their notation 1.37 30 ($1 + 37/60 + 30/3600$) would equal one and five eighths and might refer to $1^\circ\ 37'\ 30''$.

The Greeks used the Egyptian method of expressing fractions, except in astronomical computations where they used the Babylonian sexagesimal system. The Greeks used the letters of their alphabet for number symbols, for example, α, β, γ, δ . . . . κ, λ, ... for 1, 2, 3, 4, . . . 10, 20, ... etc. In using the unit fraction only the denominators were written and an accent mark was used to indicate that a fraction was meant, for example, $\gamma' = 1/3$.

The Romans used a duodecimal system of fractions due to the fact that their systems of money, weights, and measures were mainly duodecimal. The Roman abacus was arranged for a quinary-decimal system of whole numbers and a duodecimal system of fractions.

The earlier concepts of fractions were based on the idea of aliquot parts. The concept of a fraction as a number did not arise until later. The fraction has been called the first artificial number, the whole numbers being regarded as natural numbers. The whole numbers grew up in the natural process of counting individuals, necessary in the most primitive condition of human society, but the fractions were devised for the purpose of measurement. Euclid treated fractions in the form of ratios and regarded them solely as ratios. Diophantus (§17) was the first to consider fractions as numbers, thus extending the numbers from the system of positive integers to include all rational positive numbers.

Decimal fractions were first systematically treated by **Simon Stevin** in 1585. Stevin wrote them as follows: 6 (0) 2 (1) 8 (2) 4 (3)[16] for 6.284. Another notation, used later, would express the same number as $6|\underline{284}$. The decimal point (first a comma) was used by some early in the seventeenth century but was not generally adopted for another century.

15. The Signs of Operation. The signs +, −, ×, ÷, √, =, may be considered as belonging either to algebra or arithmetic. They are fundamental in arithmetic. With 5000 years of mathematical history reaching back into the infant days of civilization, we are amazed to find that these simple symbols, which to us seem indispensable, have been in use but a little longer than 400 years. Some of them were first printed less than 300 years ago. Mathematicians of the middle ages were digging deep into the theory of equations and revising Euclid's masterpiece of geometry, but they wrote an equation like this example from the "Ars Magna de Rebus Algebraicis" of Jerome Cardan:[17]

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[16] The small numbers were each enclosed in a circle.

cubus p 6. rebus aequalis 20,

for

$$x^3 + 6x = 20$$

The signs + and − were first printed in an arithmetic published by **Johann Widman** in 1489. He used them to denote excess and deficiency. The plus sign is supposed to have grown out of an inverted t, ʇ, which had been used for plus and which was an abbreviation of the Latin word **et** (and). The Latin words plus (more) and minus (less) were long in use to indicate addition and subtraction and their initial letters **p** and **m** succeeded them for a time even after the publication of Widman's arithmetic. The equation just quoted from Cardan shows the use of **p** for plus.

In **Robert Recorde's** "The Whetstone of Witte", published in 1557, we first find the use of the equality sign, =. Recorde said:[18]

> And to avoide the tediouse repetition of these woordes: is equalle to: I will sette as I do often in woorke vse, a paire of parallelles, or Gemowe[19] lines of one lengthe, thus: ═══, bicause noe .2. thynges can be moare equalle.

The cross as a symbol for multiplication began to be used in England soon after 1600. It is commonly credited to **William Oughtred** (1631). It doubtless grew out of the custom of indicating division of fractions by what was known as "cross multiplication", thus:

$$\frac{4}{5} \div \frac{6}{7} = \frac{28}{30}$$

was written

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[17] Smith, *History of Mathematics*, II, 462.
[18] Smith, *History of Mathematics*, II, 412.
[19] Twin.

$$\frac{4}{5} \times \frac{6}{7} = \frac{28}{30},$$

or by some

$$\frac{6}{7} \times \frac{4}{5} = \frac{28}{30}.$$

The sign did not become popular until the nineteenth century. The dot as a symbol for multiplication originated about the same time as the cross, being first used by **Harriot** (1631), then by Leibniz.

The sign ÷ is first found in print in an algebra (1659) by **Johann Rahn**. John Pell, who sometimes gets credit for this, assisted in publishing a translation of an algebra by Rahn.

The Hindus used the abbreviation **ka (karana,** root) when they wished to indicate the process of square root. In Europe the Latin word "radix" was abbreviated by a sign which was a combination of capital R and an x like the abbreviation for recipe used on a physician's prescription. This was not until the latter part of the fifteenth century. A dot preceding the number was used by some to indicate square root. **Christoff Rudolff** and **Michael Stifel** were among the first, early in the sixteenth century, to use the familiar symbol, √, Rudolff stating that it was an abbreviation for the "radix quadrata".

The symbols mentioned in this section were not the only ones used for the various processes. They are merely indicative of the fact that the symbols we use did not all spring suddenly from the brains of men credited with their first publication. Many and varied were the abbreviations used by mathematicians from the time of Diophantus (third century after Christ) who is mentioned in §17 as one of the founders of abbreviated algebra. The art of printing had a great deal to do with the general standardization of signs and symbols in the fifteenth and sixteenth centuries.

16. The Origin of the Name, Algebra. The word "algebra" is derived from the Arabic word "Al-jebr", or Al-jabr (both forms being quoted by historians). In the ninth century an Arab author **Al-khowarismi,** published an algebra under the title, **Al-jebr w'almuqabala,** which means "restoration and reduction" and refers to the transposition of terms from one side of an equation to the other and the reduction of the equation to a simpler form by the collection of terms. Thus the name grew out of the most important topic treated in the subject; namely, the solution of equations.

17. The Use of Literal Symbols for Numbers. Two distinct steps were made in the advance from the era of verbal solutions to the present symbolic method by means of equations. The following quotation from Heath[20] summarizes the situation perfectly:

> In his work *Die Algebra der Greichen* Nesselmann distinguishes three stages in the evolution of algebra. (1) The first stage he calls "Rhetorical Algebra" or reckoning by means of complete words. The characteristic of this stage is the absolute want of all symbols, the whole of calculation being carried on by means of complete words and forming in fact continuous prose. . . . (2) The second stage Nesselmann calls the "Syncopated Algebra," essentially like the first as regards literary style, but marked by the use of certain abbreviational symbols for constantly recurring quantities and operations. . . . (3) To the third stage Nesselmann gives the name of "Symbolic Algebra," which uses a complete system of notation by signs having no visible connection with the words or things which they represent, a complete language of symbols, which entirely supplants the "rhetorical" system, it being possible to work out a solution without using a single word of ordinary language with the exception of a connecting word or two here and there used for clearness' sake.

Diophantus of Alexandria, who lived during the latter part of the third century after Christ, is generally credited with being the "father" of syncopated algebra. He used the sign ⋔ , like an inverted and truncated Greek letter Ψ, for subtraction. The signs used for the unknown quantity in an equation were two; one like an inverted h, **ɥ** , the other like an ordinary **s**. Diophantus, who is the first person recorded as having used a single symbol for the unknown, defines the unknown quantity as "containing an indeterminate or undefined multitude of units", calling it **arithmos**[21] (number). To indicate addition he wrote the addends side by side. Instead of merely writing the constant term in the equation as such by omitting the symbol for the unknown, he preceded the constant term by the symbol $\overset{|}{\mathrm{M}}$. Initial letters were used freely as abbreviations.

The Hindus, a few centuries later than Diophantus, used a syllabic abbreviation illustrated by the following examples. **Ya** (**yavattavat**, the however much) stood for the unknown. Multiplication was indicated by **bha** (**bhavita**, the **product**). **Pha** (**phalan**) was used for the equality sign.

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[20] *History of Greek Mathematics,* II, 455.

[21] Heath, *History of Greek Mathematics,* II, 456.

Symbolic algebra grew up in the sixteenth and seventeenth centuries. **Francois Viete** (1540-1603), more generally known as **Vieta,** though not the very first to use letters as symbols for general numbers, extended their use to such an extent that he is justly called the "father" of symbolic algebra. He used the vowels, A, E, I, O, and U for unknowns. Being a wealthy man, he spent a considerable sum of money having his books published and broadcasted. On account of this method of dissemination of these new ideas the use of symbolic algebra spread rapidly and the symbolic notation was practically fixed in its present form before the end of the seventeenth century. **Descartes** (1637), who is mentioned later in connection with exponents (§20) and graphs (§22), originated the custom of representing variables by means of the last letters of the alphabet, x, y, z, etc., and constants by the first letters a, b, c, etc.

18. Negative numbers. Diophantus (§17) was probably the first to make a distinction between positive and negative numbers. He used names which meant a "forthcoming" (positive) and a "wanting" (negative). He stated further that "a wanting multiplied by a wanting makes a forthcoming and a wanting multiplied by a forthcoming makes a wanting". He used no sign for plus, indicating addition by juxtaposition, as was mentioned in §17. He wrote all the positive terms of an expression first and then had all the negative terms following the negative sign[22].

A Hindu astronomer-mathematician, **Bhaskara**, (about 1150 A. D.) seems to have been the first to take note of the negative roots of an equation. Diophantus had solved equations for positive roots only. It is certain that the Hindus recognized negative numbers. Cajori[23] says:

> They (the Hindus) brought out the difference between positive and negative quantities by attaching to one the idea of "possession," to the other that of "debts." The conception also of opposite directions on a line, as an interpretation of + and — quantities, was not foreign to them. . . . Thus Bhaskara gives $x = 50$ and $x = -5$ for the roots of $x^2 - 45x = 250$. "But", says he, "the second value is in this case not to be taken, for it is inadequate; people do not approve of negative roots." Commentators speak of this as though negative roots were seen, but not admitted.

The Hindus placed a dot over a number to indicate that it was a negative.

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[22] Heath, *History of Greek Mathematics*, II, 459.

[23] *History of Mathematics*, p. 93.

Bhaskara stated that a negative times a negative is equal to a positive and that the square root of a positive is plus or minus.

Because it is an artificial number the negative was not readily adopted. It remained for Descartes with his idea of opposite directions as exemplified in the Cartesian system of coordinates to furnish the visual representation of the negative number and thereby give it a character that finally admitted it to the society of numbers. In contrast with the time of Bhaskara, people of the eighteenth and nineteenth centuries "approved" of negative roots.

19. **The Equation.** The first known record of an equation solved is contained in an old papyrus manuscript copied by **Ahmes** (or **Ahmose**), an Egyptian, about the seventeenth century before Christ from an earlier manuscript written in the nineteenth century before Christ. This old manuscript is in the Rhind collection in the British Museum and is commonly known as the **Rhind Payrus** or the Ahmes Papyrus. An English translation has been recently made by T. Eric Peet[24]. The title of the papyrus is[25]: "Rules for enquiring into nature, and for knowing all that exists, every mystery, every secret."

In this document, the oldest "book" on mathematics in the world, the unknown quantity is called hau (heap). The first problem reads (according to Peet) "A quantity whose seventh part is added to it becomes 19". We would state the problem as an equation thus:

$$x + \frac{1}{7} x = 19$$

The Egyptian method of solution of the problem was distinctly rhetorical and amounted to the same as a solution of the proportion 8 : 7 :: 19 : x. The Egyptian gets one-seventh of the quantity equal to 2¼ + 1/8 and the quantity equal to 16½ + 1/8. Note the adherence to unit fractions.

Little improvement was made in either the notation or the method of solution of equations until the time of Diophantus (§17). He solved the general linear and quadratic equations involving one variable (but for positive roots only) and recorded the partial solutions of one cubic. Among the equations given by Heath[26] as solved by Diophantus are the following:

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[24] Peet, *The Rhind Mathematical Papyrus*, University Press, of Liverpool.

[25] *Ibid*, p. 33.

[26] *History of Greek Mathematics*, II, 464.

$$x^2 = 4x - 4;\ \text{therefore } x = 2.$$

$$84x^2 + 7x = 7;\ x = \tfrac{1}{4}$$

$$630x^2 - 73x = 6;\ x = 6/35.$$

Diophantus also solved pairs of quadratic equations in two variables and indeterminate equations of the first and second degree. Some have given him credit for first solving indeterminate problems of the second degree, but Pythagoras and others preceded him in this.

Diophantus and Bhaskara (§18) each knew a general formula for the solution of the quadratic equation. Diophantus solved the equation only for the positive roots, however, and Bhaskara only for positive and negative real roots, neither of them recognizing a general solution that would indicate complex roots.

Special forms of cubic equations were solved by the Greeks, but the solution of the general cubic was not accomplished until the sixteenth century. The first solution is credited to **Cardan** (1501-1576) who is accused of stealing it from Nicolo, who was more generally known as **Tartaglia** (stammerer), according to the following story. Tartaglia, a teacher in Venice, had become interested in the solution of the cubic through a problem solving contest in 1535 in which he vanquished his rival by solving the proposed list of thirty problems in two hours, the rival not being able to solve any of Tartaglia's problems. His fame spread and it soon became known that he had solved the general cubic equation. Cardan promised that he would keep the method secret if Tartaglia would impart it to him. Cardan broke his promise and published the solution in his "Ars Magna" in 1545. It has since borne the name of "Cardan's Solution".

Ferrari (1522-1565), a pupil of Cardan's, discovered a general solution of the equation of the fourth degree. This was also published in the "Ars Magna". No real comprehension of the complex roots of an equation was had by any of these men.

Abel (1802-1829), a Norwegian, proved that it is impossible to solve the general equation of higher degree than the fourth, that is, it is impossible to express the roots in terms of the coefficients by means of numbers formed by rational operations and the extraction of roots.

20. Exponents. The delay in the adoption of the simple modern exponential notation ranks with that of the use of the zero symbol. Rhetorical algebra, according to Nesselman (§17), was succeeded by syncopated algebra about the time of Diophantus (third century after Christ) and that, in turn, by symbolic algebra, introducted by Vieta in the sixteenth century. Yet we find Vieta and others of his time using the complete words "cube" and "square" to indicate powers. We even find x^2 and x^3 written xx and xxx on the mathematics papers of the students of our American universities during the eighteenth century[27].

Oresme (1360) had the fundamental idea of the exponential notation but did not develop a popular form of indicating exponents. **Nicolas Chuquet,** as early as 1484, attempted to abbreviate the notation to indicate powers. He omitted writing the base, $12x^5$ times $10x^3$ equals $120x^8$ being written 12^5 times 10^3 equals 120^8. **Simon Stevin** (1548-1620), a Belgian, used as a symbol for the unknown a circle and indicated the power by a small number placed within the circle. He used some fractional exponents.

It was **Descartes** (1637) who introduced the present notation for exponents, and Newton (about three decades later) who made a complete extension of the idea to fractional and negative exponents.

21. Irrational and Complex Numbers. As the integers failed to serve all man's needs for simple quantitative expression, causing the development of fractions, so the fractions failed to fill the need for adequate expression of the results of geometric measurement, and irrational numbers were devised. The Pythagoreans (§42) are credited with the discovery of lines whose measures are irrational. The proof that $\sqrt{2}$ can not be expressed as a rational fraction was known long before the time of Euclid who gives the proof presented in §26. This proof is believed to be the one used by the Pythagoreans, but the Pythagorean policy of secrecy makes it impossible to verify the supposition. The old story of the punishment by drowning of the member of the Pythagorean school who divulged the secret proof of the irrationality of $\sqrt{2}$ is classed among the unprovable legends. The Greeks kept the idea of the irrational confined to geometry and the measure of lines. They failed to consider it as a pure number.

Little improvement was made on the idea of irrationals until modern times. It remained for Dedekind, Cantor, and others make a logical definition of irrational number (§28) in the second half of the nineteenth century.

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[27] See "Introduction of Algebra into American Schools in the Eighteenth Century," by Lao Genevra Simons; published by the U. S. Bureau of Education as Bulletin, 1924, No. 18.

The complex number owes its existence in mathematics to the same cause as does the negative; namely, the equation. Both negative and complex roots were noticed for centuries before they were interpreted. Because of the apparent lack of application to material things they were both regarded as fictitious numbers. The name "imaginary", which still clings to the complex number, gives evidence of how it was first regarded. In the attempt to solve the general equations of the second, third, and fourth degrees the square roots of negative numbers were constantly recurring. Gauss, in the early part of the nineteenth century, adopted the notation $i = \sqrt{-1}$ and devised the geometric representation of the complex number as we have it today (§33).

22. **Graphs.** When most of the older teachers of this generation first studied algebra there was no trace of graphs in the text. Many of the younger teachers were taught by teachers who omitted the chapter on graphs. (Let us hope it was not because they did not understand it.) Every text in algebra published in the last ten years treats of the subject, many in such a manner that it can not easily be omitted, and practically all junior high school texts now make it an important part of the subject matter of the seventh and eighth grades. Indeed, the use of graphs in the newspapers and magazines is so prevalent, and a pupil's need of ability in the future to interpret graphs is so certain, that the school owes it to him to teach him this.

Many commercial forms of graphical representations could hardly be classified under the same head as the graphs of continuous algebraic functions studied in algebra twenty years ago. Nevertheless, many graphical representations of statistics show how one variable quantitiy is dependent on another, and uses two intersecting axes. In such case it has as its parent the idea of coordinates as presented by Descartes in 1637, when he published "La Geometrie" and therein laid the foundations for analytic geometry. The system of coordinates originated by him is the one, familiar to all of us, known as Cartesian coordinates (from his Latin name, Cartesius).

The nearest approach to the use of coordinates (as we know them) before Descartes was made by **Apollonius** about two centuries before Christ. From his monumental work on conic sections it is evident that he had within his grasp the fundamental ideas of coordinates. The lack of development of algebra, which among the Greeks was apparently neglected for the sake of geometry, might have been the cause of his failure to develop a coordinate system; for he studied the conics by means of the relation of the "ordinates" (parallel lines crossing a diameter of the conic).

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CHAPTER III.

FUNDAMENTAL CONCEPTS OF THE NUMBER SYSTEM

23. Introduction. Some illustrations were given in Chapter I of mistaken ideas of what is meant by mathematics. Many errors are made in the teaching of algebra because of the lack of knowledge on the part of the teacher of the fundamental concepts underlying the subject. Errors have been written into elementary texts because of the same lack of knowledge on the part of the authors. A notable example of this is the attempted proof of the statement, $x^\circ = 1$, which can be considered logically only as a definition. The absurdity in the attempted proof is the assumption that the exponent zero has all the characteristics of a positive integral exponent which was defined as a number that indicates "how many times the base is used as a factor".

Before studying methods of presenting the subject matter of algebra to our students let us examine some of the primary concepts underlying the subject. We begin with the study of the number system.

24. Natural Development of a Number System. The idea of number is first born in the mind of a child when he comprehends the difference between one thing and two things. He may be able to say the words one, two, three, etc., up to ten and not have a conception of number. Eventually he has a conception of the values of the numbers up to ten, though the real worth of the larger values may be vague, and can probably count, parrot like, beyond ten—perhaps to one hundred. These numbers gradually force themselves upon his mind because they are needed to count the things of his experience, his fingers, his toys, and other things. He calls these numbers by the same names as we do but if there were no names he would invent names. Since there was in the life of each of us, in all probability, a pause in our advance in the learning of numbers when we reached ten, we may say that our first number system was composed of the first ten integers.

There was one boy who, when he had learned how to combine the words to produce names for numbers beyond one hundred, took pride in counting to a high mark by cumulative effort. Remembering where he had quit at bedtime the night before, he would in rare periods of idleness take up the count where he had left off and go a few hundreds further until he ran up into the thousands. His intelligent comprehension of the values of numbers was confined to numbers less than one hundred. His experience is no doubt universal and we may say that the second number system in the experience of the average individual is composed of the positive integers up to a point of indefiniteness limited by the inability to name them further, of which comparatively few are clearly comprehended.

Later, in school, the child learns the meaning of fractions, and his third number system includes all the positive rational numbers. When he studies algebra he extends his number system to include negative numbers and somewhere along the way irrationals are included. Further yet he goes and imaginary, or complex, numbers are added. So we see that the number field is always increasing. In each stage of the development more numbers can be named than can actually be realized in the experience of the individual and the number system seems indefinitely large. Five hundred is "infinity" to a child of four.

It is a familiar theory that a child in his mental development lives over again the experiences of the human race in its struggle for enlightenment. It is certainly true in the case of the development of the number system. The need for new numbers has preceded their development in every case. In terms of the meaning and use of the numbers of any established system those of the next higher are "imaginary". Each new field is arbitrarily developed and named when the need for it arises.

The system of positive integers is used to count whole things It is unthinkable (imaginary) to say there are twenty and one-half sheep in a pen. The fraction was not needed by our prehistoric, nomadic ancestor to count his sheep or his camels. If a farmer wishes to announce his average yield (per acre) of wheat the fraction is needed and twenty and one-half bushels is not a nonsensical expression at all; but minus twenty bushels of wheat is meaningless. On the other hand, minus twenty degrees as a statement of the temperature is clearly understood. Illustrations showing how complex numbers may be used in practice and have simple interpretations are given in §36.

Let us now turn from the natural development of numbers by the human race and consider the subject from the standpoint of a mathematical science. We will make the idea purely abstract and remove it from all its applications.

25. The Nature of a Logical Development of the Number System. It is not claimed that the following treatment is a complete logical development nor that it contains a minimum list of independent postulates. It is an illustration, true as far as it goes, of the nature of such a development and is purposely made elementary[28].

We say "four men", "four pencils", etc. Each expression has a clear and unmistakable meaning and carries a mental picture with it. What do we mean when we merely say "four"? It is an abstract idea. We can not comprehend it unless we illustrate it by making it apply to concrete things. It will apply equally as well to one group of things as another. It can be thought of then as that attribute of a group of distinct things which is independent of the things themselves or of their order of arrangement within the group.[29] Note the abstractness of the idea of pure number and compare the quotation from Poincare (§3). Again, if two things can be put into a one-to-one correspondence, that is, if there is an element of one group for every element of the other, and conversely, we say that the two groups have the same cardinal number.[30] Suppose this idea is applied to the points of two line segments of unequal lengths. These points can be put into a one-to-one correspondence in the following manner. Connect the ends of the line segments (making a quadrilateral) and extend the connecting lines until they meet. Through this point of intersection draw lines intersecting both of the given segments. Every line thus drawn pairs off two points, one on each line segment, thus putting the points into a one-to-one correspondence. There are apparently as many points on the short segment as the long one, hence the points are innumerable for the two lines segments will not exactly coincide. This gives us a concept of the so-called infinite number. It is not a definite number at all.

Let us assume that we have defined the fundamental operations of addition (symbol $+$), multiplication (symbol $\times$), and order (symbol $<$ or $>$). Let us make some assumptions which will determine the character of the numbers we mean to use.

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[28] For a logical development see J. W. Young, *Fundamental Concepts of Algebra and Geometry,* or J. W. A. Young, *Monographs on Modern Mathematics,* No. IV.

[29] Fine, *The Number System of Algebra,* p. 3.

[30] Young, *Fundamental Concepts,* Lecture VI.

If a, b, c, and d are any numbers of our system:

Postulate 1. $a + b$ always exists in the system and is a unique number.[30a]

Postulate 2. $a + b = b + a$. (This is called the commutative law of addition.)

Postulate 3. $(a + b) + c = a + (b + c)$. (This is called the associative law of addition.)

Postulate 4. $a \times b$ always exists in the system and is a unique number.

Postulate 5. $a \times b = b \times a$. (Commutative law of multiplication.)

Postulate 6. $(a \times b) \times c = a \times (b \times c)$. (Associative law.)

Postulate 7. $a \times (b + c) = a \times b + a \times c$. (Distributive law.)

Postulate 8. If $a = b$ and $c = d$; then (1) $a + c = b + d$, and (2) $a \times c = b \times d$.

Try the following systems of numbers on the above postulates, one by one, and discover which will not satisfy them:

1. The system of positive integers.
2. The system of integers from one to one hundred.
3. The system of odd integers.
4. The system of even integers.

The positive integers operate according to the above list of postulates. If we limit ourselves to the integers from one to one hundred we see that addition and multiplication are not always possible (for example, 58 × 74 is not in the system) so Postulates 1 and 4 will not hold. All but Postulate 1 are valid for the system composed of odd integers. All of them are satisfied by the system of even integers.

We see then that whether or not certain laws are true concerning our numbers depends on what the number system contains. If we were an untutored race of savages knowing how to count only to one hundred, we could not add two such numbers as 50 and 70 for we could not express the sum in terms of anything we should know.

If we restrict ourselves to these assumptions in building up our number system we can not combine our numbers in any way but by addition and multiplication. We would therefore restrict our number system to the positive integers. We shall set down other assumptions (postulates) that will extend the system.

[30a] This means that it shall always be possible to perform the operation of addition on any two numbers and secure a result which belongs also to the system. For illustration, suppose a child of four can count only to ten and is taught to add such combinations as $2 + 3$, $4 + 5$, etc. Suppose he tries to add $7 + 6$. Within his knowledge of numbers the sum does not exist.

Postulate 9. There exists one and only one number z of the system such that $a + z = a$. (Thus we define zero and add one more number to the system.)

Postulate 10. For every number a there exists a number a' such that $a + a' = z$. (This defines the number a', which we may call the opposite, or **negative,** of a. By this postulate we have doubled our number system.

Postulate 11. If a and b are fixed numbers of the system there exists one and only one number x such that $a + x = b$.

A process is here defined which is inverse to addition. We shall call it **subtraction.** This postulate is not always true of the system of numbers defined by the first eight postulates; namely, the positive integers. For example, when $a = 7$ and $b = 3$, x could not be expressed. The child studying arithmetic in the elementary school will inform you with confidence that there is no number which may be added to 7 to obtain the sum 3. How then does such a number come to exist? It is postulated, as is being done here. For the child it is done when he first studies algebra; for the human race it was done when necessity demanded it. Using the symbol **minus** ($-$) we say that $x = b - a$.

Postulate 12. If and b are two fixed numbers of the system, and a is not z, there exists a unique number x such that $a \times x = b$.

This defines a process inverse to multiplication which we shall call **division.** Since by the assumption (postulate) it must always be possible to divide, we have added rational fractions to our system. Note that this assumption was not always true of the system of positive integers (for example, when $a = 5$ and $b = 3$).

If $a = b$, the number x defined by this postulate will be called the **unit** of the system (symbol, 1).

These twelve assumptions define for us the rational numbers, that is, the numbers obtained by a finite number of operations of addition, subtraction, multiplication, and division. Postulates 1 and 4 make the system infinite. Postulates 2, 3, 5, and 6 avoid troublesome numbers. Without Postulate 7 we could not develop our processes of multiplication of polynomials (or numbers in arithmetic with two or more digits) and could not factor.

From these postulates may be derived our laws of operation for rational numbers. One such proof will be given as an illustration.

Theorem I. Adding a negative number a' is equivalent to subtracting the corresponding number a.

Our problem is to find the sum of a' and b.

Let $x = b + a'$.

then $x + a = b + a' + a$ (Postulate 8.)

and $x + a = b + z$, (Postulate 10)

whence $x + a = b$. (Postulate 9)

Then x can be found by the process of subtracting a from b, or

$x = b - a$. (Postulate 11)

$b + a' = b - a$. (Postulate 8)

All of the operations and principles of elementary algebra can be justified on the basis of these postulates. Let this word be said here: This is **not** the way to teach these operations to a group of high school students. More will be said about this in a later chapter on methods.

26. The Need for Irrational Numbers. The definition of power grows out of the application of Postulate 4 if we repeatedly use the same number as an operator. Then we immediately consider the inverse operation and find that the result can not always be expressed in our system of rational numbers. For example, if $x^2 = 2$, what is the value of x?

We will insert here (for there may be some who have not seen it) the Euclidean proof of the irrationality of $\sqrt{2}$, that is, a proof that $\sqrt{2}$ can not be expressed as a rational fraction, or, that it can not be secured by the operations of addition, subtraction, multiplication, and division.

Let us suppose that $\sqrt{2} = m/n$, where m and n are prime to each other (merely reducing the fraction to its lowest terms) and prove it impossible. If $\sqrt{2} = m/n$, then $m^2 = 2n^2$.

(1) m and n can not both be even numbers because they are prime to each other.

(2) m and n can not both be odd for then m^2 would be odd and could not be equal to the even number $2n^2$.

(3) The same argument opposes m being odd and n even.

(4) Finally, if m is even and n odd let $m = 2k$. Then we have $4k^2 = 2n^2$, or $2k^2 = n^2$, where n^2 is odd and $2k^2$ even.

We are led to an absurdity in every one of the four possible cases and must conclude that $\sqrt{2}$ is not equal to m/n, or that there exists no rational number which will, when squared, produce the number 2.

If we have had no past experience with any other than rational numbers we can say that $\sqrt{2}$ is inexpressible, "imaginary", or unreasonable (irrational). But our concrete applications arise and demand such a number. If the length of a side of a square is unity, the length of the diagonal is $\sqrt{2}$. The diagonal of a square is just as definite and real as a side, and can be constructed with the tools of elementary geometry, the ruler and compasses. This number must be provided for in our assumptions concerning numbers or else our system will be limited in its possible use. We must create, logically, a number whose square will be 2.

27. Sets of Numbers. Upper Bounds. Consider the following sets (collections) of numbers:

$$(1)\quad 1,\ 1\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1\frac{7}{8}\quad .\ .\ .\ .\ .\ .$$

$$\text{or, } 2-1,\ 2-\tfrac{1}{2},\ 2-\frac{1}{2^2},\ 2-\frac{1}{2^3},\ .\ .\ .\ .\ .\ .\ 2-\frac{1}{2^{n-1}}$$

(2) 1, 1.4, 1.41, 1.414, 1.4142, ... continuing to closer and closer approximations of $\sqrt{2}$ as secured by application of the familiar process of square root.

The limit approached by the terms of set (1) is 2. What is the limit approached by the terms of set (2)? Let us use the notation $a_1, a_2, a_3, \ldots\ldots a_n$ for any general set like those above and let the symbol [a] be an abbreviation for the set. We proceed as follows:[31]

A set of numbers [a] is said to have an **upper bound** M if there exists a number M such that there is no number of the set [a] greater than M. For example, there exists no upper bound for the set of positive integers, 1, 2, 3, . . . , for there is no limit to the size of the integers in the number system. Again, any number greater than 2 is an upper bound of the set (1) for no term of the set is greater than 2; and 2 is itself an upper bound.

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[31] Adapted from Veblen and Lennes, *Infinitesimal Analysis*, Wiley, 1907.

We see also that 1.5, 1.42, 1.415 are among the upper bounds of the set (2).

The term "upper bound" does not define any certain number. There are many upper bounds of a given set, if any. Sometimes there seems to be a least one as in set (1). In this set if we take n large enough we can make the nth term differ from 2 by less than any assigned positive quantity. We shall call 2 the **least upper bound** of set (1); for if there be another number less than 2 which is presumably an upper bound we can make the n-th term of set (1) approach nearer to 2 than the difference between 2 and the so called upper bound which is smaller than 2 by some finite quantity. A set may or may not reach its upper bound. In set (2) we can not name a fixed rational number as the least upper bound, for as we extend the number of decimals in the terms of our set we can find continually decreasing upper bounds without limit.

28. Definition of Irrational Number.

Postulate 13. If a set [a] of rational numbers having an upper bound has no rational least upper bound, then there exists one and only one number B such that:

(1) $B > a_n$, where a_n is any number of the set [a].

(2) $B < M_n$, where M_n is any rational upper bound of the set [a]. This number B is called the least upper bound of set [a] and is evidently not rational. We call it an **irrational** number. This definition in different language is due to Dedekind[32].

Referring to the set (2) of the preceding section for a concrete illustration, $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, $a_4 = 1.414$, etc., and $M_1 = 2$, $M_2 = 1.5$, $M_3 = 1.42$, $M_4 = 1.415$, etc. The definition as given postulates, or assumes, for this set a number B which lies between the increasing a_n and the decreasing M_n.

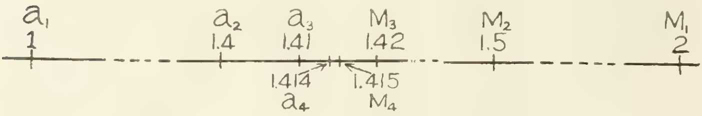

FIGURE 3

29. The Geometric Analogy. Let us suppose that our numbers can be put into a one-to-one correspondence with the points on a line. Suppose we assign a point for every rational number. Would we exhaust our points? No, for it is evident that however finely we divide our fractions there would exist between any two points as represented by rational fractions other points (say, between the point represented by 2 and that represented by 2.0000000000001 would lie 2.00000000000001 and others). In other words, there would be "gaps" in the line. It would not be continuous. Consider the number $\sqrt{2}$. It lies somewhere between 1.4 and 1.5, for $(1.4)^2 < 2 < (1.5)^2$. We can restrict its location further. It lies between 1.41 and 1.42, between 1.414 and 1.415, and so on. There is a point on the line for $\sqrt{2}$ as surely as the diagonal of a square can be constructed. None of the points which correspond to irrationals like $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, etc., agree with any of the points corresponding to the rational integers or fractions. The totality of all the points for both rational and irrational numbers makes a continuous line. This completes what is known as the system of real numbers. In further remarks on the number system no attempt will be made to proceed in a logical manner, but it is aimed to illustrate the nature and meaning of other numbers than reals.

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[32] Dedekind, *Essays on Number*, (a translation).

30. Outline of the Number System.

I. Algebraic numbers
  1. Real
    (a) Rational
      Positive and negative, or
      Integral and fractional
    (b) Irrational (called algebraic irrationals)
      Positive and negative
  2. Complex (sometimes called imaginary)
II. Transcendental numbers (also irrational)

31. Why Have "Imaginary Numbers"? In the solution of quadratic equations we are brought face to face with the problem of the square root of a negative number. For example, the quadratic $x^2 + x + 1 = 0$ has the roots

$$x = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \quad x = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}$$

Either we must say that some quadratic equations can not be solved, or we must include the square root of a negative among the numbers used to express the values of the roots. This dilemma contributed directly to the development of the so-called imaginary numbers.

We also notice that the introduction of negative numbers gives us two square roots of every positive number (a positive and a negative) instead of the one (positive) that we know in arithmetic. There is, however, only one real cube root of a number, positive if the number is positive and negative if the number is negative. By analogy we should expect three cube roots. If the equation $x^2 = 1$ has two solutions, why does not the equation $x^3 = 1$ have three solutions? Are there three answers to the question, "What is the cube root of unity?" We can answer the question by writing the equation $x^3 - 1 = 0$ and solving it. If we factor the expression we get $(x - 1)(x^2 + x + 1) = 0$, whence

$$x = 1, \quad -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \quad -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}$$

Each of these expressions raised to the third power gives unity. Thus, by using complex quantities we write down all three of the cube roots of unity. Similarly, the solution of the equation $x^4 = 1$ yields the results 1, -1, $\sqrt{-1}$, and $-\sqrt{-1}$, all of which are fourth roots of unity.

Another reason for the existence of complex, or "imaginary", quantities is their practical usefulness, though this reason for existence did not arise until the theory of complex quantities had been developed. This is an illustration of the situation referred to in §3 where the application was found long after the theory had been developed as an abstract problem in pure mathematics. These applications will be considered after we examine the nature of complex quantities from a purely mathematical standpoint.

32. Definition of Complex Quantities. We may define a complex quantity as a "couple" of real quantities which stand for different things. Examples of number couples (or pairs) that stand for different things are: 7 cows, 6 horses; 8 pens, 5 pencils; 2 "ones", 3 i's. Let us completely remove the notion of concrete things and use an abstract notation, say (7, 6), (8, 5), (2, 3) for the number couples suggested above. Let us call the two real quantities a and b the first component and the second component[33] of the complex quantity (a, b).

33. Vector (Geometric) Representation of Complex Quantities.

The numbers of arithmetic are scalar. In their applications they refer to magnitude alone and answer the questions, "How many?" and "How much?" They have only one attribute from the standpoint of measurement and that is size. The expression $m < n$ means that m is smaller in magnitude than n. The symbol $<$ refers to order of magnitude.

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[33] A complex quantity may have three or more components (it is then usually called hypercomplex), or we may say that it is merely an aggregate of several kinds of simple quantities (real numbers) any one of which may be absent or zero.

When we study mechanics we meet quantities that can not be fully understood or applied unless we know a direction which is inseparable from the quantity itself. The effect of a force of ten pounds can not be considered if we do not know the direction in which the force acts. The same is true for a velocity. Such quantities are represented by vectors. A vector is a directed quantity and can be completely determined only by means of two given quantities, its scalar magnitude and its direction. The notion of directed quantities is introduced into algebra along with negative numbers. In that case there are but two directions, the positive and the negative. The expression $m < n$ means "m precedes n on the scale."

Let (a, b) represent a typical complex quantity (according to the notation of the last section), where a and b are arbitrary real quantities (the components). Our system of numbers so defined can be put into one-to-one correspondence with the points on a plane, for each couple (a, b) uniquely determines a vector from the origion O to the point P (Fig. 4) with the rectangular coordinates (a, b). The system of real numbers is included within the system of complex quantities (cases where $b = 0$).

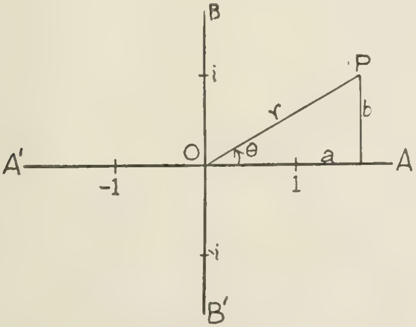

FIGURE 4

This representation is credited to Gauss, a German mathematician of the nineteenth century. It is the familiar illustration seen in any college algebra.[34] The theory of complex quantities may be developed entirely apart from this geometric illustration, though it is used to clarify some of the ideas.

34. Addition of Complex Quantities. It becomes necessary to make a set of assumptions covering the whole field of possible operations with complex quantities.[35] These processes must be so defined as to harmonize with the assumptions concerning real numbers. We shall not form a complete set in this discussion but will illustrate the method of procedure on the basis of the definition of complex quantity given in §32.

If we have 7 horses and 3 cows (7, 3) in one field, and 4 horses and 2 cows (4, 2) in another field, the total number in both fields is 11 horses and 5 cows (11, 5). That is, if $\alpha = (7, 3)$ and $\beta = (4, 2)$, $\alpha + \beta = (11, 5)$ by definition. In general, if $\alpha = (a_1\ b_1)$ and $\beta = (a_2, b_2)$, we define the sum as

$$\alpha + \beta = (a_1 + a_2, b_1 + b_2).^{36}$$

35. Multiplication of Complex Quantities. If c is a real number, $c(a, b) = (ca, cb)$, by definition. In particular, $-1(a, b) = (-a, -b)$. We agree that $-1(a, b) = -(a, b)$, and that $(0, 0) = 0$. If $(a, b) + (x, y) = 0$, we say (x, y) is the negative of (a, b), that is, if $(a + x, b + y) = 0$, $x = -a$, $y = -b$.

It is necessary to define what is meant by the product of two complex quantities, say $\alpha\beta$, where $\alpha = (a_1\ b_1)$ and $\beta = (a_2, b_2)$. Let us note the equation $x = ab$, where x, a, and b are real numbers. To solve this equation we try to find the number which bears the same relation to b that a does to unity: $a/1 = x/b$. Let us define multiplication of complex quantities so that in the solution of $\xi = \alpha\beta$ we find that number which has the same relation to β that α has to the unit vector. We will use the vector representation (Fig. 4) to assist in this. Let α be the complex quantity represented by the vector OP whose scalar magnitude is r and direction-angle is θ. The operation on the unit vector necessary

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[34] By this method the complex quantity is represented by $a + bi$; or by $r(\cos\theta + i \sin\theta)$, where r and θ represent the scalar magnitude and direction-angle, respectively, of the quantity (see Fig. 4).

[35] Consult Young, *Monographs on Modern Mathematics,* No. IV, "The Fundamental Propositions of Algebra," by Edward V. Huntington.

[36] If we use the familiar notation of the usual college algebra we express it as follows: $\alpha = a_1 + ib_1$, $\beta = a_2 + ib_2$, and

$$\alpha + \beta = (a_1 + a_2) + i(b_1 + b_2).$$

to produce α is a stretching in the ratio r:1 (that is, the scalar magnitude of the unit vector is multplied by r) and a rotation counter-clockwise through the angle θ. Suppose now that β has the scalar magnitude r_1 and the direction-angle θ_1. To get the product $\alpha\beta$ we must stretch, or multiply, the scalar magnitude r_1 by the factor r and rotate the vector through the angle θ. The vector of the product then has the scalar magnitude rr_1 and the angle $\theta + \theta_1$.

It is obvious from Fig. 4 that $a = r \cos \theta$ and $b = r \sin \theta$. If we take into consideration in our definition these relations then we may define the quantity $\alpha = (r_1 \cos \theta_1, r_1 \sin \theta_1)$ and $\beta = (r_2 \cos \theta_2, r_2 \sin \theta_2)$. In general, then, we can define multiplication as follows:

$$\begin{aligned} \alpha\beta &= (r_1 \cos \theta_1,\ r_1 \sin \theta_1)(r_2 \cos \theta_2,\ r_2 \sin \theta_2) \\ &= [r_1 r_2 \cos(\theta_1 + \theta_2),\ r_1 r_2 \sin(\theta_1 + \theta_2)] \end{aligned}$$

If we expand these expressions according to the formulas of trigonometry we obtain

$$\begin{aligned} \alpha\beta = r_1 r_2 \cos \theta_1 \cos \theta_2 - r_1 r_2 \sin \theta_1 \sin \theta_2, \\ r_1 r_2 \sin \theta_1 \cos \theta_2 + r_1 r_2 \cos \theta_1 \sin \theta_2) \end{aligned}$$

Note that in the vector-representation the product vector has a direction-angle equal to the sum of the angles of the factors.

If we use the notation $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$ (remembering the relation $a_1 = r_1 \cos \theta_1$, etc.) the product becomes

$$\alpha\beta = (a_1 a_2 - b_1 b_2,\ b_1 a_2 + a_1 b_2)$$

Using this form we obtain the product $(c, 0)\ (a_2\ b_2) = (ca_2, cb_2)$, which harmonizes with the definition of this product given at the beginning. In particular, $(-1, 0) = -1$; and $(0, 1)(0, 1) = -1$. Hence $(0, 1) = \sqrt{-1}$, which corresponds to the arbitrary unit i. Since $(a, b) = (a, 0) + (0, b) = a(1, 0) + b(0, 1) = a + bi$, we can use the notation $a + bi$ for a complex quantity. It is a more convenient notation.

We noted that the direction-angle of a product is the sum of the direction-angles of the factors. The direction-angle of a negative real number represented on the complex plane is 180°. The product of two such numbers will have the angle 360°. This (or 0°) is the angle for a positive real. Hence it follows that the product of two negative numbers is a positive number. For example:

$$-3 = (3 \cos 180°, 3 \sin 180°).$$
$$-4 = (4 \cos 180°, 4 \sin 180°).$$

The product is written

$$(3\cdot 4 \cos 360°, 3\cdot 4 \sin 360°) = (12, 0) = 12.$$

If we use the other form of notation, we have:

$$-3 = 3(\cos 180° + i \sin 180°)$$
$$-4 = 4(\cos 180° + i \sin 180°)$$

The product is written

$$3\cdot 4(\cos 360° + i \sin 360°) = 12(1 + i\cdot 0) = 12$$

It is as impossible to explain this to a high school pupil as it is foolish to teach him a pretended proof. Since he obviously can not be taken through the fundamental postulates and the consequent development of the theorems concerning the rules of operations, would it not be wiser and more in keeping with the spirit of mathematics to assume the law of signs at the outset?

36. Applications of Complex Quantities. Wherever vectors are used the principle of complex quantities is involved and the notion of complex quantities can be used.

In the theory of electricity (alternating currents) the amount of the current is expressed as a vector. Two currents have different phases (expressed in degrees) and magnitudes (expressed by scalar numbers). In this situation complex quantities can be, and are, used. They are also used in connection with voltages. The problem may be solved without the use of complex quantities but the adoption of the notation greatly simplifies and shortens the solution.

The following illustration taken from Wentworth's "College Algebra", (1902), p. 528, shows not only how complex quantities may have material interpretation but how negative and fractional numbers may be "imaginary" in certain cases.

(1) Two clocks begin striking at the same moment; one of the clocks strikes six strokes more than the other, and the number of strokes struck by one of them is double the square of the number struck by the other. Find the number of strokes struck by each clock.

(2) The temperatures indicated by two thermometers differ by 6° and the number of degrees in the temperature indicated by one of them is double the square of the number of degrees in the temperature indicated by the other. Find the temperature indicated by each.

(3) Two men start to walk from the same place at the same moment; at the end of an hour they are 6 miles apart, and the number of miles one of them has walked is double the square of the number of miles the other has walked. Find the number of miles each has walked.

Each of these problems gives rise to the equations

$$y = 2x^2 \qquad \text{and} \qquad y - x = \pm 6.$$

By substitution we obtain the equation $2x^2 - x = \pm 6$.

$$(2x^2 - x - 6)(2x^2 - x + 6) = 0, \text{ whence}$$

$$x = 2,\ -1\tfrac{1}{2},\quad \tfrac{1}{4} + i\sqrt{47/4},\quad \tfrac{1}{4} - i\sqrt{47/4}$$

$$y = 8,\quad 4\tfrac{1}{2},\ -23/4 + i\sqrt{47/4},\ -23/4 - i\sqrt{47/4}$$

In the first problem we say that the first pair of values 2, 8 are the only ones that satisfy the equation and that the others are impossible ("imaginary"). In the second problem both the first and second pairs of values are permissible, but the complex roots of the equation are out of the question. In the third problem all of the solutions of the equations may be interpreted as satisfying the conditions of the problem, depending on the direction each man traveled. If both traveled in the same direction, the first pair give the lengths of their journeys; if they traveled in opposite directions the second pair expresses their journeys, the signs determining the directions. If they traveled in diagonal directions as indicated by Fig. 5 the actual number of miles traveled was $\sqrt{3}$ and 6 respectively, and they were six miles apart at the end of the hour. The vectors representing the actual journeys are determined by the third pair of values according to the usual conventions with regard to complex quantities. The representation of the fourth pair is similar.

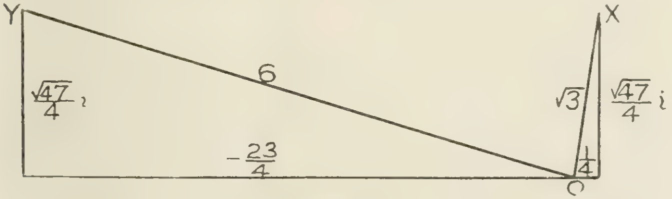

FIGURE 5

37. Transcendental Numbers. The numbers considered up to this point are known as algebraic numbers. Any **algebraic num-**

ber may be the root of an equation $f(x) = 0$, where $f(x)$ involves the variable x in a finite number of operations of addition, subtraction, multiplication, division, and root extractions. Numbers that can not be roots of such equations are called transcendental. The most familiar transcendental number is π (=3.1416, nearly), which expresses the ratio of the circumference of any circle to its diameter. It is an interesting fact that such a commonly used number, expressing such a simple relation in our practical experience, can not even be classed among the algebraic irrationals. It has no place in the system of algebraic numbers. We can construct no line by means of ruler and compasses whose length is exactly π units as we may construct a line whose length is, say, $\sqrt{2}$ units. The proof that it is transcendental[37] (first by Lambert and Legendre in the eighteenth century) puts it beyond possibility of construction as a line segment with ruler and compasses and stops (or should stop) forever all attempts to "square the circle".

38. Quaternions. The theory of quaternions was developed by W. R. Hamilton in the first half of the nineteenth century. A quaternion involves four units, 1, i, j, k with the relations $i^2 = j^2 = k^2 = ijk = -1$, from which follows $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. Or, we may say that a quaternion has four components (see foot note §34). Notice that the commutative law does not hold for multplication. We can not say of any two quaternions Q_1 $(= a_1 + b_1i + c_1j + d_1k)$ and Q_2 $(= a_2 + b_2i + c_2j + d_2k)$ that $Q_1 Q_2 = Q_2 Q_1$. Quaternions have many applications.

39. Conclusions. If these considerations concerning the source, and possible extension, of our number system teach us nothing more it shows clearly how arbitrary and abstract the system really is. We can not look upon the laws of operation, the commutative law, for example, as being anything else than agreements at the outset that the numbers we use shall be such as will obey those laws. We should not concern ourselves about proving those laws to pupils in the high school. Illustrations may be aptly used in every case to show how such a rule (say, the subtraction of a negative equivalent to the addition of a positive) ought to be expected if the results of using the newly learned (negative) numbers in algebra are to have a sensible in-

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[27] See Young's *Monographs,* No. IX, History and Transcendence of π," by D. E. Smith, or see §45.

terpretation. If the pupil understands these laws as arbitrary "rules of the game" and can come to a feeling (after some use of them) that they produce reasonable results he will have a more accurate conception of the real situation than if made to believe that he "proves" them.

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CHAPTER IV

A BRIEF HISTORY OF GEOMETRY.

40. The Origin of Geometry. No written record can reach back to the origin of geometry. The sources of geometry were in the world before the existence of man. His need of the knowledge of these facts brought them to light. Since the earth is so nearly a sphere, he was compelled to learn the properties of a spherical surface in order to navigate successfully upon the ocean. It was necessary for him to have a method of laying off a perpendicular in order to survey his fields. He learned how to find areas and volumes in order to buy and sell land and the products of the land. Little by little, as the need for them arose, man discovered the geometric relations we know so well.

The oldest record of man's development and use of geometry is found in the Rhind Papyrus, (§19) which may be called a mathematical "handbook" that was in use 3700 years ago. In the Rhind Papyrus we find the formulas for the areas of the rectangle, triangle, trapezoid, and circle. The formulas (except the one for the area of the rectangle) are only approximately correct. For example, the area of a circle is given as $64\ d^2/81$, where d is the diameter. This assumes a value of 3.1605 (nearly) for π. The area of a trapezoid is given as equal to half the product of the sum of the bases by one of the nonparallel sides. If the nonparallel side happens to be perpendicular to the parallel sides the formula is correct. The area of a triangle drawn with base four and one side ten is given as $(\frac{1}{2}\times4\times10)$ which is correct if the ten-unit side is perpendicular to the base. Some insist that the poorly drawn figure represents an isosceles triangle, in which case the formula is incorrect.

Herodotus and other Greek historians state that the Greeks learned geometry from the Egyptians. The principle use made of geometry by the Egyptians was in the realm of mensuration. The need of the Egyptians for relations of measurement between geometric figures grew out of the annual inundation of the Nile valley, which made it necessary for the Egyptian "surveyors" to reestablish boundary lines, and also out of their extreme care in the construction of their temples. Thus the ancient progenitors of two of our modern applied sciences, surveying and architecture, which we realize are dependent upon geometry, had much to do with forcing the first stages of its development. References to the Egyptian surveyors (called "rope-stretchers") on ancient inscriptions antedate the Rhind Papyrus. There seems to be no doubt that they knew that a triangle the lengths of whose sides are three, four, and five units has a right angle opposite the longest side. They stretched a rope around three pegs so as to produce such a triangle and, consequently, a much needed right angle.

Simultaneous with the development of simple practical geometry in Egypt was the development of the same subject in Babylonia and the other oriental nations, though the historical record does not extend so far into the past as in Egypt. As early as 1500 B. C. the Babylonian value of π was 3, which value was also used by the Hebrews.[38]

The historical period in China begins somewhere between 1100 B. C. and 800 B. C. It is fairly certain that the Chinese made astronomical observations earlier than 2000 B. C. This involved a knowledge of mathematics. The earliest Chinese classic that can be called mathematical was in existence earlier than 1105 B. C. for it tells of the wonderful prowess of Chou-Kung who died 1105 B. C. Among the things about geometry known by Chou-Kung was the relation between the sides of the "3-4-5" triangle.

In the sacred writings of the Hindus, probably about five centuries before Christ, we find a statement of the Pythagorean Theorem, but there is no evidence of a proof. There is also a statement of special case of it; namely, that the diagonal of a square is the side of a square of twice the size of the first square. The Hindus also seem to have had "rope-stretchers" for they constructed by means of cords what they called "rational rectangles", in which the diagonals as well as the sides are rational. Besides the triad (3, 4, 5) they seem also to have known right triangles with the sides (5, 12, 13), (7, 24, 25), and (12, 35, 57). They also knew that "the diagonal of a square produces an area double" (of the original square), and could construct a square equal in area to a given rectangle.

41. Thales. Thales (about 624-547 B. C.), listed by Plutarch as one of the Seven Wise Men, may properly be called the Father of Greek Geometry. It was he who took the first step beyond the inductively discovered rules of mensuration and other practical geometry of the Egyptians and began to make of geometry a deductive science. He beat the Egyptians at their own game of practical measurement when on one of his visits to Egypt he astonished the king by measuring the height of a pyramid by means of its shadow. This implies the theory of similar triangles. The earliest historical mention of this feat, however, only credits him with measuring the shadow of the pyramid when the shadow of a body equals the height of the body.[39] It has been noted by some that such a simple method as this could not have astonished the Egyptians when one considers their advancement at this time (and even earlier) along the lines of practical measurement, hence he must have known something of the theory of similar triangles.

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[38] 1 *Kings*, vii. 23, and 2 *Chronicles*, iv. 2.

It has been asserted that Thales was the first to discover the following general theorems, for some of which there is no record of proof by him:

(1) A circle is bisected by its diameter.

(2) The angles at the base of an isosceles triangle are equal.

(3) If two straight lines intersect, the vertical angles formed are equal. (Thales did not prove it.)

(4) An angle inscribed in a semicircle is a right angle.

(5) If two triangles have two angles and one side equal, respectively, the triangles are congruent.

The assumption that he knew the theorem stated last is based on the story that he was able to measure the distances of ships from the shore, which must involve some form of the congruence theorem as stated. He probably used right triangles, making his observations from the top of a vertical tower first in the direction of the ship and then along the edge of a level stretch of beach.

Some interesting stories of the life of Thales can be found in Smith's "History of Mathematics", Vol. 1, Chap. III.

42. **Pythagoras.** One of the most noted of the Greek Mathematicians was Pythagoras (about 572-497 B. C.). At Crotona in Southern Italy he assembled about him a group of mathematicians who formed a secret brotherhood, the mysterious badge of which was a five-pointed star. The veil of secrecy was so closely drawn about them that it is impossible to identify the discoverer of any of the theorems accredited to the Pythagoreans.

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[39] Heath, *History of Greek Mathematics,* I, 129.

It is not certain that Pythagoras himself was the one who first proved the famous theorem which bears his name; that is, "The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs", though we would like to believe that he did. The proof used by the Pythagoreans is believed to be the one wherein a perpendicular is dropped from the vertex of the right angle on the hypotenuse (c) making segments m and n on the hypotenuse adjacent to the legs a and b, respectively, forming similar triangles. From this we have

$$a^2 = cm$$

$$b^2 = cn$$

$$a^2 + b^2 = c(m + n) = c^2$$

As a natural outgrowth of their study of this theorem the Pythagoreans gave considerable attention to irrational numbers. They investigated commensurability and first proved that the diagonal of a square is incommensurable with its side (§26).

The Pythagoreans have an indisputable claim to the discovery and proof of the theorem concerning the sum of the angles of a triangle for the general case, though the fact was probably known by the Egyptians for special forms like the equilateral and isosceles triangles. Proof was made by means of a line through a vertex parallel to the opposite side. They also discovered and proved the theorems about the sums of the interior and exterior angles of a polygon. Their ability to prove the angle-sum theorems demonstrates that they knew the properties of parallel lines.

They had a theory of proportion partly developed and were aware of the properties of similar figures.

They knew how to divide a line into mean and extreme ratio and were the first to construct a regular pentagon. In their study of regular figures they discovered that only three can be made to fill a plane space without overlapping; namely, the equilateral triangle, the square, and the regular hexagon.

They originated the subject of equivalent areas and the transformation of an area of one form into another of a different form.

More detailed information about Pythagoras and the Pythagoreans can be found in Heath's "History of Greek Mathematics," Vol. 1, which is the source of the information given here.

43. Contributions of Other Greek Geometers. The two centuries between Pythagoras and Euclid (500-300 B. C.) were fruitless in the production of new propositions in geometry. The most noted geometer of this period is **Plato,** [430-349 (?) B. C.], whose supreme contribution to the science of geometry was to place it on a sound logical basis. He considered mathematics from the viewpoint of philosophy. He scorned the idea of its use for practical purposes and extolled its virtues as a mind trainer. Over the door of his Academy (school of philosophy) was written, "Let no one destitute of geometry enter here". He is reputed to have said, "God eternally geometrizes." He insisted on rigorous methods of proof, with exact and consistent definitions, and was the first to give definite statements of axioms and postulates as such. He is also regarded as the originator of the analytic method of proof (§56).

A friend of Plato was **Archytas**, whose time is only approximately known but believed to be the first of the fourth century, say about 400-350 B. C. It was doubtless because of his interest in the problem of the duplication of the cube (§48), which he solved by means of mean proportionals, that he was led to the development of the following theorems which are believed to have been proved by him.[40]

1. If a perpendicular is drawn to the hypotenuse from the vertex of the right angle of a right-angled triangle, each side is the mean proportional between the hypotenuse and its adjacent segment.

2. If the perpendicular from the vertex of a triangle is the mean proportional between the segments of the opposite side, the angle at the vertex is a right angle.

3. The perpendicular is the mean proportional between the segments of the hypotenuse.

4. If two chords intersect, the rectangle of the segments of one is equivalent to the rectangle of the segments of the other.

5. Angles in the same segment of a circle are equal.

6. If two planes are perpendicular to a third plane, their line of intersection is perpendicular to that plane and also to their lines of intersection with that plane.

Archytas was also among the first to apply mathematics systematically to mechanics. It is recorded that he made a mechanical wooden dove that would fly.[41]

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[40] Smith, *History of Mathematics*, I, 85.

[41] Heath, *History of Greek Mathematics*, I, 355.

During the century preceding the time of Archytas and Plato, that is, the fifth century B. C., **Hippocrates** and **Oenopides** made valuable contributions to geometry. It is told of Hippocrates that he was a bankrupt merchant who turned to geometry. References are made by early Greek writers of mathematical history to his "Elements" of Geometry, a book like Euclid's and preceding it by two centuries, which is lost. He is the first to be credited with an attempt to organize the propositions of geometry into a logical sequence in textbook form. Though complete and authentic records are hard to find it is considered safe to assume that he was the first to develop extensively the theorems on circles. He was probably led to this by his interest in the problem of "squaring the circle". It is certain that he knew and probable that he discovered and proved the following theorems:

1. In the same circle or in equal circles equal chords subtend equal arcs, and conversely.
2. Angles inscribed in the same or equal segments are equal.
3. The sum of the opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.
4. An angle inscribed in a semicircle is a right angle.
5. The ratio of the areas of two circles is equal to the ratio of the squares of their diameters. [That he proved this is regarded as uncertain, Archimedes claiming that Eudoxus (about 375 B. C.) was the first to prove it.]
6. Similar segments of circles are to one another as their bases. (Regarded as certain.)

His discoveries in connection with the problem of squaring the circle and duplication of the cube will be found in §§45, 47.

Oenopides was an astronomer. The staff he used to place in a vertical position for the purpose of casting shadows so as to determine the time of day was called a gnomon. He developed a method of constructing a "gnomon-wise" line, as he called it, from a point to a line. Thus we have the first record of the construction of a perpendicular from a point to a line. He is also given credit for first inventing a method of constructing on a given line an angle equal to a given angle. His method was to complete a triangle with the given angle as one of its angles and to construct by use of the three known sides a congruent triangle. The same method was used by Euclid.

44. Euclid. The time and place of the birth and of the death of the most famous of ancient mathematicians are unknown. But for what he left the world as a monument to his patient, untiring, scholarly research he would now be unknown. Some scholars have written volumes but only to those who search the shelves of the libraries are their names familiar. This man gave the world one volume for which his name has been familiar to the schoolboys of generations upon generations. To be the author of a best seller has always meant fame, but the reputation of writing a text that has been used 2200 years, with various revisions, has fallen to but one man in the history of the world—Euclid. Books on plane geometry published more than fifty years ago bore the single title—Euclid.

All that is recorded concerning the time of his life is that he lived during the reign of the first Ptolemy (306-293 B. C.). His nationality is uncertain. It is supposed that he studied at Athens. He founded a school at Alexandria, Egypt. He wrote on other subjects than geometry but his "Elements", as his great work on geometry is called, was his masterpiece.

The "Elements" consisted of thirteen so-called "books", each of which was written on a separate parchment-roll, and represented as nearly as was humanly possible all the essential, known theorems on geometry and allied subjects up to his time. Some of the theorems were doubtless original. Many of the proofs were. His masterful work however, was not what he originated but how he collected and arranged what others had proved. Earlier workmen in the centuries preceding him, even back to the building of the pyramids, had made the parts; he, the master mechanic, assembled them into a perfect scientific machine, securely riveted with the logic of Plato. So carefully selected and stated are the definitions, axioms, and postulates that this "Elements" is still the fundamental basis of our courses in plane geometry to-day.

Thus does Greece pay back to Egypt, with interest, that which had been borrowed centuries before when (according to Herodutus) Thales went down into Egypt and learned geometry.

The contents of the "Elements" by books are as follows:

I, Congruence, parallels, the Pythagorean Theorem.

II, Identities which we would now treat algebraically, like $(a + b)^2 = a^2 + 2ab + b^2$, but which were then treated geometrically; areas; the Golden Section.

III, Circles.

IV, Inscribed and circumscribed polygons.

V, Proportion treated geometrically; in part, a geometric way of solving fractional algebraic equations.

VI, Similarity of polygons.

VII-IX, Arithmetic (the ancient theory of numbers) treated geometrically.

X, Incommensurable magnitudes.

XI-XIII, Solid Geometry.

Much more could be written about Euclid. For instance, the Arabs explained that the name of Euclid (they called it Uclides) was compounded of "Ucli", a key, and "Dis," a measure, and meant the "key of geometry".[42] The story of his calling a slave to hand some pennies to a royal pupil who asked in disgust what profit he would derive from the study of geometry is found in many books, even in texts on plane geometry. We will assume that more detailed knowledge about Euclid can easily be found by any reader and save space for other topics.

45. Squaring the Circle. The problem of squaring the circle, that is, constructing by means of ruler and compasses a square equal in area to a given circle, early claimed the attention of the geometers. It followed naturally the development of methods of mensuration of rectilinear figures. We find the sequence developed in historical and logical order in the usual text in plane geometry. First the area of the rectangle, then the area of the parallelogram, triangle, and trapezoid, are found by formulas. A simple method of constructing a triangle (whose area is readily transformed into a square) equal in area to any given polygon is known. The next question that arises is: Can we find a square equal in area to a given circle?

Since the area of a given circle with radius r is found by the formula $A = \pi r^2$, the problem rests upon the value of the numinnumerable and ingenious attempts to show that a square could be found equal in area to a given circle. Later mathematicians worked to find a more nearly correct value of π, extending by laborious computation the number of known decimal places. Among those the name of Ludolph von Ceulen (1540-1610) is prominent. He extended the known value to 35 decimal places and is reputed to have spent 14 years computing the last 15 places. Because of his labors, π was for a long time called the Ludolphian number. Finally, (after the middle of the 18th century) the power of modern analysis was turned on the problem and it was proved by **Lambert** (1766) and **Legendre** (1794) that π is an incommensurable number. Later (1882) **Lindemann** proved that it is a transcendental number (§37). That settled the problem so far as mathematicians are concerned. There are still sporadic attempts by those who know no mathematics to "square the circle". They can not, or will not, understand why their methods are wrong and their efforts in vain.

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[42] Heath, *History of Greek Mathematics*, I, 355.

One illustration of the stubbornly persistent attitude of some of the so-called "circle-squarers" will be given. A business man, James Smith, of Liverpool, had some correspondence with Dr. De Morgan, one of the noted English mathematicians of Cambridge about the middle of the nineteenth century, concerning the value 3⅛ for π. Smith gave a French well digger credit for originating the value but he (Smith) insisted that it was correct. He refused to be convinced even when shown by one of De Morgan's friends that if a polygon of twenty-four sides is inscribed in a circle the ratio of the perimeter of this polygon to the diameter of the circle is 3.17, which is greater than 3.125, the ratio (claimed by Smith) of the circumference to the diameter.

Two points of interest to the student of mathematical history occur in the study of this problem; namely, the methods used in the attempt to solve the problem, and the extension of the number of decimal places in the value of the number π. We shall consider the latter first.

The earliest approximation (and poorest on record) for π was 3. It is thought that the Babylonians used this value. The Hebrews were still using it at the time of Solomon (1 Kings, vii. 23 and 2 Chronicles, iv. 2). The Egyptians had a much better approximation, $(16/9)^2$, or 3.1605 (nearly), according to the Rhind Papyrus of Ahmes. **Archimedes** (about 225 B. C.) inscribed polygons to approximate a circle, doubling the number of sides up to 96. He showed that the value of π was between 3 1/7 and 3 10/71. **Chang Hong** (A. D. 78-139), a Chinese astrologer who wrote on astronomy and geometry, is reputed to have given the value $\sqrt{10}$ for π, a poorer approximation than that known by Ahmes more than 1700 years earlier. There are recorded six different values of π given by Chinese mathematicians, the most remarkable one being given by **Tsu Ch'ung Chih** (A. D. 430-501) who showed that the value lies between 3.1415926 and 3.1415927 and who used 22/7 and 355/113 as approximate values. No closer approximation than this was discovered in Europe before the sixteenth century. Two values used by Hindu mathematicians were $\sqrt{10}$ (seventh century after Christ) and 3.1416 (time uncertain).

We will mention two of the many methods used by the ancient geometers in their attempts to construct a square equal in area to a given circle and give a detailed explanation of only one.[43]

Antiphon (fifth century before Christ) introduced the method of "exhausting" the area inside a circle by continued construction of regular inscribed polygons, each of which has twice as many sides as the one preceding. He argued that since a square can be constructed equal in area to any polygon and an inscribed polygon can ultimately be constructed whose perimeter coincides with the circle, then a square can be constructed equal to the circle. Archimedes also used the method of exhaustion in computing his approximate values for π.

Hippocrates (§43) was the first geometer to construct a square equal in area to some curvilinear figure. His plan was as follows: On the sides of an inscribed square as diameters construct semicircles as in Fig. 6. Since each smaller semicircle is equal to half the larger semicircle, the lunes AB and BC are equal to the triangles ABD and BCD, respectively. Hence a square can be constructed equal to a lune. Hippocrates' next step was to take an inscribed hexagon and show that since the semicircle on the diameter of the hexagon is equal to the sum of four semicircles on the sides of the hexagon as diameters, then the trapezoid (Fig. 7) ABCD = I + II + III + IV. Assuming that a square may be constructed equal to each of the lunes he investigated (and consequently a square may be constructed equal to the difference between the trapezoid and the sum of the

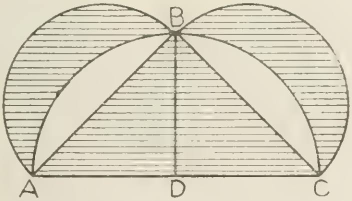

FIGURE 6

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[43] For additional information consult Heath, *History of Greek Mathematics,* Vol. I. Chap VII; J. W. A. Young, *Monographs on Topics of Modern Mathematics,* No. IX, "The History and Transcendence of π," by D. E. Smith; and Hobson, *Squaring the Circle.*

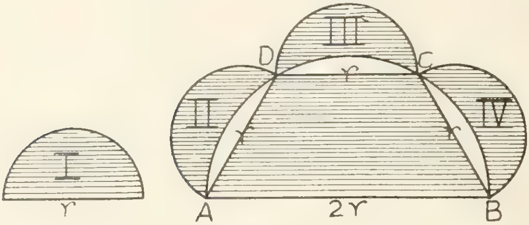

FIGURE 7

lunes) he concludes that a square may be constructed equal to the semicircle I. His was not the first nor last error made in mathematics by assuming a general truth from the results of one special case.

When attention was transferred in the seventeenth century from the problem of finding a square equal to a circle to the problem of determining the nature of the number π, better methods were developed. By the use of infinite series (could he have known about it) Ludolph von Ceulen might have saved many years labor. The value of π has been computed to 707 decimal places only 30 of which, according to the astronomer, Newcomb, "are sufficient to give us the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope".

46. Trisection of the Angle. It is supposed that this problem arose out of the attempt to inscribe in a circle a regular polygon of nine sides. The early Greek mathematicians solved the problem of inscribing the regular polygons of 4, 5, and 6 sides (and of course any multiples of these numbers). The inscription of a regular nonagon would involve the trisection of the 120-degree angle. As in the case of the problem of squaring the circle, modern analysis proves that the problem is not possible by means of Euclildean geometry, that is, by the use of the straight-edge and compasses. The following brief considerations tend to show how the problem becomes impossible[44].

Suppose we desire to find x, which is one-third of some given angle. Then 3x equals the given angle. An algebraic relation between the angles 3x and x is given by the equation:

$$\cos 3x = 4 \cos^2 x - 3 \cos x.$$

[44] For a complete discussion, see J. W. A. Young, *Monographs on Modern Mathematics*, No. VIII, "Constructions with Ruler and Compasses," by L. E. Dickson.

Since the first member of this equation is known, we have a cubic equation in one unknown, cos x, which (if we let $y = \cos x$) we may write in the form:

$$4y^3 - 3y = c,$$

where c is a known constant; namely, the cosine of the given angle which is to be trisected. The solution of this equation involves cube root and the cube root of a number can not be constructed by the methods of Euclidean geometry.

Probably the earliest method of attack on this problem was quoted by Pappus (§48), who lived and taught at Alexandria during the latter part of the third century after Christ. The author is unknown.

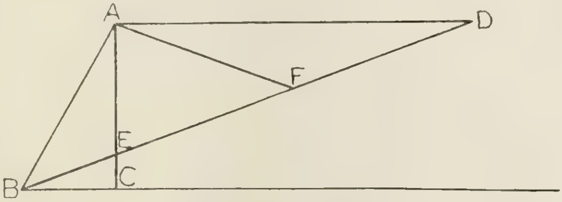

FIGURE 8

Let ∠ABC be given; draw AC perpendicular to BC; draw AD parallel to BC; let BD, intersecting AC in E, be drawn so that DE = 2AB; and let F be the middle point of DE. Then ∠ABC = 3 (∠EBC); for since triangle DAE is a right triangle, AF = EF = FD. Triangles ABF and AFD are isosceles and ∠ABF = ∠AFB = 2(∠ADF) = 2(∠EBC). The problem is thus reduced to another; namely, to draw from B a line BD to cut AC and AD in such a manner that DE = 2AB.

The latter problem was solved in more than one way, but not by any method of construction by ruler and compasses. **Nicomedes,** who lived during the middle of the third century before Christ, devised a curve called the "Conchoid of Nicomedes" by means of which the point D (Fig. 8) can be found. The polar equation of the curve is

$$\rho = a + b \sec \theta.$$

It may be defined as the locus of a point in a line which revolves on and slides in a fixed pivot so as to allow a constant portion of the line to project beyond a fixed straight line. It is possible to construct a mechanical device that can be used to trace the curve

47. **The Duplication of the Cube.** Two stories are told of the origin of this problem. One is that King Minos was disappointed with the size of a cubical tomb under construction for his son and ordered the builders to "double it but fail not to cubical form". Another is that the Athenians were promised by their oracle at Delos that they could rid themselves of a troublesome plague if they would but double a certain cubical altar, whereupon they constructed one with an edge twice as long (volume eight times the original) and the plague became worse. They appealed to the geometricians. The story is but a legend and it is suspected that the mathematicians inspired the oracle's pronouncement in order to arouse interest in the problem.

The construction involves the solution of the equation $x^3 = 2a^3$, where a and x are the edges of the original and desired cube, respectively. It is impossible by the methods of Euclidean geometry to perform the construction for the same reason as that given for the impossibility of the solution of the problem of trisecting an angle.

The problem was reduced by Hippocrates to one of finding two geometric means between two numbers one of which is twice the other, that is, to find x and y so that

$$a:x = x:y = y:2a.$$

These two equations may be written $ay = x^2$ and $y^2 = 2ax$, whence $y = x^2/a$, or $y^2 = x^4/a^2$, and $x^4 = 2a^3x$, or $x^3 = 2a^3$.

Archytas (§43) and others devised schemes for finding by mechanical construction the required value. Archytas' solution involves solid geometry. Other solutions are by means of higher plane curves (the "Cissoid of Diocles" being a notable one) as was Nicomedes' solution of the trisection problem.

48. **Post-Euclidean Geometry.** The Greeks produced very little that was original in elementary plane geometry after Euclid. There were critical comments on the "Elements", a very noted one being that of **Proclus** (A. D. 410-485) who had access to historical works on geometry now lost and whose commentary on Euclid is the principle source of information as to the history of elementary geometry before his time. Proclus and others questioned Euclid's postulates and the discussion eventually led to the development of non-Euclidean geometry.[45] After Proclus little was added to the theory of geometry for a thousand years. The Arabs translated the "Elements" (and other Greek books)

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[45] See Chapter V for a full discussion of non-Euclidean geometry.

into their own language about the ninth century and have the distinction of giving geometry back to Europe, for the first Latin translation of the "Elements" (twelfth century) was from an Arabic translation of the original, no Greek manuscripts being then at hand.

The most famous of the Greek geometers following Euclid was **Apollonius** who lived in the second half of the third century before Christ. His main contribution to geometry was a treatise on conic sections, which bears to the study of the parabola, ellipse, and hyperbola the same relation that Euclid's "Elements" does to the study of the rectilinear figures and circles. It was the first authoritative text on the subject. Apollonius was not the first to discover and study these curves, for **Menæchmus** (4th century B. C.) is reputed to have written on the subjects, though some uncertainty surrounds the records. Menæchmus is supposed to have cut a different cone with a plane to produce each curve; a right-angled cone for the parabola, an acute-angled cone for the ellipse, and an obtuse-angled cone for the hyperbola. Apollonius derived all three curves from the same cone, changing the direction of the intersecting plane for the different curves. His near approach to the discovery and use of coordinates is mentioned in §22. Little attention is now given to the methods of Apollonius in developing the theorems relating to conics. The use of analytic geometry has supplanted the ancient methods.

Proclus says that Apollonius was the first to use the method of bisection of a line segment that we usually find in our texts, that is, the method of drawing the common chord of equal intersecting circles having the extremities of the line as centers. Euclid's method of performing this construction was to construct an equilateral triangle on the line as a side and bisect the angle opposite the given line segment.

After Apollonius the most prominent writer on geometry among the Greeks was **Pappus**, who lived and taught at Alexandria during parts of the third and fourth centuries. He is noted for his commentaries on the "Elements" and the works of other mathematicians, his greatest work being the **Synagoge,** or Collection. The "Collection" is a sort of handbook or guide to Greek geometry and was meant to revive the study of classical geometry. It covers practically the whole field. In the preface to Book V of the "Collection," Pappus writes on the sagacity of the bees, mentioning among other things their shrewd use of the hexagonal form for their honeycomb cells[46]. He is supposed to be the author of some of the new theorems in the "Collection" though it has been suspected that he copied the works of others without giving credit. He shows how to inscribe seven equal regular hexagons in a circle in such a way that one is about the center of the circle, while six others stand on its sides and have the opposite sides in each case placed as chords in the circle. He shows how to construct a conic through five given points and gives also some theorems relating to triangles and circles which are seldom seen now in elementary texts in geometry.

A more familiar theorem of the type just mentioned is that of **Menelaus** (first century) which states that if the three lines forming a triangle are cut by a transversal, the product of the lengths of the three segments which have no common extremity is equal to the product of the other three. In the figure below AY·CX·BZ = CY·BX·AZ. Another theorem about triangles similar to the theorem of Menelaus is "Ceva's Theorem", named for an Italian geometer of the seventeenth century. He proved that any three concurrent lines through the vertices of a triangle divide the opposite sides so that the products of the lengths of the three segments having no common extremity is equal to the product of the other three. Yet another theorem showing the trend of modern geometry is the famous "nine-point circle" theorem the discovery of which is probably due to more than one mathematician working independently. It is stated thus: Let the altitudes of a triangle be drawn (meeting in a point) and let the middle points of the segments of the altitudes adjacent to the vertices of the triangle be found. These three bisection points, the feet of the three altitudes, and the middle points of the three sides all lie on a circle. Many other theorems of pure geometry have been proved by mathematicians of the last four hundred years using modern theories of involution, anharmonic ratio, reciprocal polars, homology, perspective, and projective geometry. The most

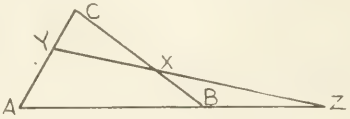

FIGURE 9

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[46] See Heath, *History of Greek Mathematics*, II, 389, 390.

important development in modern geometry is the theory of coordinates and the development of analytic geometry as introduced by Descartes in 1637.

49. Solid Geometry. Solid geometry was developed simultaneously with plane, maintaining the lead in the amount developed. No record is found as to the origin of the customary theorems about planes, perpendiculars, and other lines in space previous to the "Elements". As in plane geometry, the first developments arose out of practical necessity, volumes of solids being first investigated.

Democritus (5th century B. C.) is credited with first stating that a pyramid is one-third of a prism having the same base and altitude, and that a cone is one-third of a cylinder having the same base and altitude. The credit for first proving these two theorems goes to Eudoxus, who lived during the first half of the fourth century before Christ.

The Pythagoreans were first on record to study the five regular solids, the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. This was doubtless an outgrowth of their study of the angular magnitude about a point. **Theætetus** (about 415-369 B. C.) wrote on the same five solids and Euclid in Book XIII uses the proofs of Theætetus.

Archimedes is noted among other things for his treatise on the sphere and cylinder and so proud was he over his theorem that the volume of a sphere inscribed in a cylinder is equal to two-thirds the volume of the cylinder that he requested that the figure be cut in his tombstone. It was this figure that enabled a Roman officer to find and identify the neglected grave after his death. Archimedes first proved that the area of a sphere is $4\pi r^2$.

50. Conclusions. The developments recorded in this chapter are neither complete nor exhaustive. Only principal facts are recorded. Every teacher should do further reading in some standard history of mathematics. It is clearly seen from the data given that logical, demonstrative geometry came last, not first. Most of the theorems of geometry were discovered inductively, many of them out of the everyday experiences of ordinary people. It was the function of the mathematicans first to generalize the apparent facts that were experienced in special cases (like the 3-4-5 case of the Pythagorean Theorem), then to establish proofs for the general theorems, and finally to organize the whole in a logical, scientific manner, basing the proofs necessarily on undefined terms and assumptions of facts verified by common experience.

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Hobson, Squaring the Circle.
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Young, Monographs on Topics of Modern Mathematics.
See also the list at the end of Chapter II.

CHAPTER V.

NON-EUCLIDEAN GEOMETRY

51. The Fifth Postulate of Euclid. Broadly speaking, non-Euclidean geometry is any geometry that is not Euclidean. The term as usually considered applies to two certain types of geometry not Euclidean, that is, two geometries that substitute another postulate for the fifth postulate of Euclid.

In laying the logical foundation for the "Elements" it was necessary for Euclid to have certain unproved propositions (see §4). Those relating particularly to geometry he called postulates —"necessary conclusions". Those relating to magnitudes in general he called axioms "that which is thought worthy". His assumptions were those which the consensus of opinion would approve as being true. Any logical development of geometrical propositions based on the assumptions of Euclid is called Euclidean geometry.

Following Heath's[47] translation of Heiberg's edition of Euclid we set down Euclid's fifth postulate.

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, produced indefinitely, meet on that side on which are the angles less than two right angles.

From this he deduced the following:

> Through a given point one and only one straight line can be drawn which will be parallel to a given straight line. (I. 31).

In the modern texts as most of us have known them this proposition is given as the postulate, or axiom, of parallels from which the fifth postulate is deduced as a theorem. Other theorems depending for their demonstration on this assumption are those relating to the sum of the angles of a triangle and properties of similar figures. Whatever be the order selected, there must be some assumption concerning parallels as a starting point.

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[47] Heath, *The Elements of Euclid,* Vol. I.

The earliest known commentaries on Euclid took exception to this postulate. **Proclus,** of Athens, in the fifth century declined to consider it a postulate and attempted to prove it. **Saccheri,** in the opening years of the eighteenth century attempted to prove it by assuming it false and trying to establish its truth by finding a contradiction in his development of geometry. He found no logical contradiction, which is enough to suggest that possibly a logical system of geometry might be established with some other postulate of parallels as one of the foundations. This is just what **Lobachewski** did. In 1829-35 he published several papers on geometry in which he assumed that through a point more than one line can be drawn parallel to a given line and proved, among other things, on this assumption that the sum of the angles of a triangle is less than two right angles. **Gauss** and **Bolyai** at about the same time were working independently along the same lines. In 1854 **Riemann** published a geometry in which the assumption is made that through a point no line can be drawn parallel to a given line. From this assumption follows the theorem that the sum of the angles of a triangle is more than two right angles. The Lobachewskian geometry is also called hyperbolic, the Reimannian, elliptic.

The greater part of the theorems of Euclidean geometry are also true in either of the two systems of geometry mentioned above. All theorems whose proof in Euclidean geometry depend on the postulate of parallels are necessarily replaced by others.

The term "Non-Euclidean Geometry" is generally used to mean a geometry which substitutes another postulate of parallels for Euclid's fifth postulate. Let us consider more in detail what some of these assumptions mean.

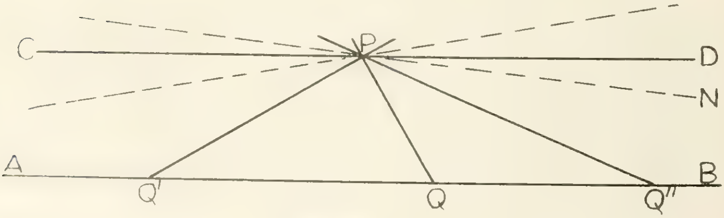

FIGURE 10

Given a line AB and a point P without the line. Let a line PQ rotate about P in a counter-clockwise direction. Its intersection point with AB moves away to the right as in Fig. 10 above. Assuming that the rotation can be continued until the lines do not intersect at all we say the line PQ is then parallel to AB. Let PN represent this position. Now if the line had been rotated clockwise until such a condition of non-intersection at the left was obtained (PM), would the lines PM and PN be one and the same straight line? Can one be sure that as the line PN leaves the line AB at the right it simultaneously touches it at the left? Or may there be positions in between these limiting cases? Euclid assumed only one non-intersecting position, CD. Lobachewski assumed two non-intersecting positions (as PM and PN) making an angle which may contain many non-intersecting lines. Riemann assumed no non-intersecting position, that is, the rotating line PQ intersected AB at the left before leaving it at the right.

One may ask, "Are these non-Euclidean geometries mere speculations that are not true?"

52. Can we Depend on Our Intuitional Judgment for Absolute Truth about Space Relations?

In seeking an answer to this question let us consider the following illustrations.

A basin of water sits in absolute stillness. What is the form taken by its surface? We say that the water will form a level surface and we may immediately jump to the conclusion (without reasoning) that the surface is a plane. We know that it is a portion of a spherical surface the radius of which is the radius of the earth.

What is the shortest distance between two places in the United States, say Chicago and New York? We often say that it is as the crow flies, which is supposed to mean in a straight line. The space of our existence is mainly on the surface of the earth and for that reason our impromptu ideas of straight-line distances are frequently at fault as we forget that a straight-line distance should cut through the earth, the chord of the arc of a great circle on the surface which we sometimes sub-consciously think of as a straight line. Suppose we were two-dimensional creatures living on a spherical surface and having no conception of a third dimension. To us then the only sensible shortest distance between two points would be the arc of a great circle on our spherical space (surface).

If we were limited to a small section of that surface we could have a geometry like our own plane geometry. (In ordinary surveying the earth's surface is now treated as a plane and the principles of plane trigonometry used.) If some master mind among our two-dimensional fellow creatures should reach out and assume that our space was curved and that our straight lines were really curved out beyond the limit of our experience, each returning to its starting point, we would declare it incredible. But such an assumption is exactly analogous to that of Einstein concerning our own three-dimensional space. The space of the Einstein theory is a curved space in which the geometry of Riemann is valid.

A postulate of Euclidean geometry omitted by Euclid from his list, but implied in his proofs, is that a figure may be moved about in space without change of shape or size. This is called the postulate of superposition. The question of whether or not it is true, as well as whether or not any of the postulates are absolutely true, is merely a question of defining the kind of space in which we are making a geometry. Suppose a two-dimensional space shaped like the shell of an egg and inhabited by two-dimensional beings who, naturally, could consider only two-dimensional figures in geometry. Could a figure on the large end, or side, be moved around the small end and still retain its shape and size? Obviously not. In such a space there could be no similarity of figures as we know them in plane geometry.

One of the statements in the theory of relativity is that bodies change size when moving through space. Any one accepting the theory of time and space as proposed by Einstein must omit the superposition "axiom" from his geometry .

Poincare (§3) has described a space, or world, of three dimensions in which non-Euclidean geometry would seem to be the real geometry.

53. Poincare's Non-Euclidean World. Suppose a world enclosed in a three-dimensional space whose boundary is a spherical surface of radius R.

Suppose that in this space the temperature varies from its maximum value at the center to absolute zero on the boundary; and at any distance, r, from the center is proportional to R^2-r^2.

Suppose all bodies have the same coefficient of expansion and that the linear expansion is proportional to the absolute temperature. Bodies would attain their maximum size at the center and become infinitesimal as they approached the boundary of the space.

Suppose that a body changing its position changes its size instantaneously, thus keeping itself in thermal equilibrium with its environment.

Under the third supposition the boundary of the space would appear to an inhabitant of the space to be at an infinite distance, for if he were to approach the boundary his steps would become shorter (due to his fast decreasing size), approaching infinitesimal distance, and it would seem to require an infinite number of steps to reach the boundary.

A very pertinent fact about the condition of an inhabitant of the space is that he would be unconscious of his change in size because his "yardstick" would decrease, or increase, in the same proportion as his body as he moved about and he would measure as many feet tall in one place as another.

Finally, suppose that light travels through media of different refractive indices such that the index of refraction is inversely proportional to R^2-r^2. Under this assumption light would travel in circle arcs. Then a person walking toward a light, keeping the light directly in front of him at all times would follow the arc of a circle. He would be sure that he approached the light by the shortest path.

Professor J. W. Young ("Fundamental Concepts", Lecture II) has devoted some space to this subject in his book. The following developments are parallel to his statements.

If in this space an inhabitant wishes to go from point A to point B in the shortest time by walking at a fixed rate, that is, taking a fixed number of steps per minute, it is a fact that he would arrive at B in less time (fewer steps) if he traveled along the arc of a circle orthogonal to the spherical boundary of the space than if he went along what we would call a straight line. This can be proved and is due to the fact that his steps would be lengthened enough along that part of the arc nearer the center of the space (due to his increases in size as he approached the center) to more than make up for the extra distance (as we would call it) along the arc.

An inhabitant of such a space would unconsciously (intuitionally) call these circle arcs shortest distances, or possibly "straight lines". There could be only one of these between two points, which is proved on the following page.

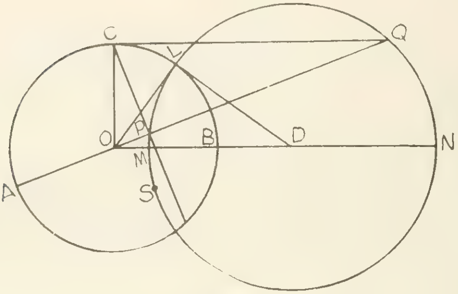

FIGURE 11

Let the circle ABC (Fig. 11) represent a cross section of our non-Euclidean space determined by some orthogonal circle through the point P on the diameter AB. Let PC be perpendicular to AB and let the tangent to the circle at C cut the diameter AB, extended, in Q. Any circle through P and Q cuts the circle ABC at right angles and every circle through P cutting the circle ABC at right angles will go through Q. From elementary geometry it is easy to verify the following equations:

$$(OC)^2 = OP \cdot OQ = OM \cdot ON = (OD - LD)(OD + LD)$$
$$= (OD)^2 - (LD)^2.$$

Therefore $(OD)^2 = (OL)^2 + (LD)^2$, and the angle OLD is a right angle. The converse statement is easily established.

Suppose we have two points P and S in this space. Since any orthogonal circle ("shortest distance") through P must go through its corresponding point Q, and since three points determine a circle, only one circle orthogonal to the circle ABC can be drawn through the points P and S. Therefore P and S determine a shortest distance.

Let us now consider Euclid's fifth postulate in this imaginary space. We will call the shortest distances straight lines.

Let the line l cut the bounding surface in A and B (Fig. 12). Let P be a point not on the line.

Suppose some line through P intersects the line AB at a point between A and B. Now let the line rotate about P in a clockwise direction until its point of intersection with AB approaches B. In the limiting position PB will seem to an inhabitant of the space never to meet AB as he can not go out there to investigate and as far as he can see or measure there is some distance be-

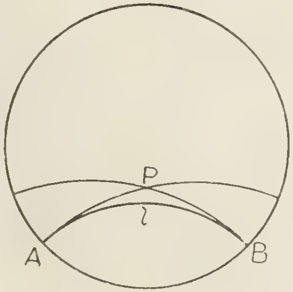

FIGURE 12

tween them. On the other hand, if we rotate the line through P in a counter-clockwise direction it will approach a limiting position PA which makes it seem again to be a "parallel" line to AB as before. In Euclidean geometry it is assumed that the two positions PA and PB are the same and only one line is parallel to AB through P. It is evident that in this new space there are two distinct positions as the pairs of points, P and A, P and B, determine two distinct straight lines (so called). It is also evident that there would be other lines through P not meeting AB, lines lying within the angle between PA and PB.

The question might at once be raised that this hypothetical situation is far removed from the space of our experience. Let us suppose our earth to be near the center of such a space with a very large, but not infinite, radius. The only circle arcs representing shortest distances which we could experience would be practically straight lines as we know them, for the nearer the point P is to the center the larger are the radii of the orthogonal circles through P, approaching infinity as a limiting value, in which case the circles would become straight lines through the center. The size of our earth is trivial compared to the measured distances out into space as computed by the astronomers. The angle between the two limiting positions PA and PB could be so small as to be impossible of measurement within the limits of instrumental error. In other words, we may never know whether there is just one, or more than one, line through a point parallel to a given line. If we assume only one we get Euclidean geometry with the sum of the interior angles of a triangle equal to two right angles, and other familiar theorems. If we assume more than one we get a non-Euclidean (Lobachewskian) geometry with many theorems different. Our trigonometry and other subjects which provide formulas for computation have been developed on the assumptions of Euclidean geometry, but might have been built on other assumptions without any appreciable difference in computed values as we know them. We can never know which is absolutely true and, as far as mathematics is concerned, do not need to know. To quote again from Poincare:

> Une geometrie ne peut pas etre plus vraie, qu'une autre; elle peut seulement etre plus commode. Or la geometrie euclidienne est et restera la plus commode. (One geometry can not be more true than another; it can only be more convenient. Now Euclidean geometry is, and will remain, the most convenient.)

A system of geometry is dependent logically on the undefined terms and the unproved propositions postulated in the beginning. Geometers claim nothing more than that these are assumptions.

54. Conclusions. What then does it profit a teacher if he gain a whole world of geometries and lose faith in the geometry of his fathers as a book of absolute truth?

A very important, if not the most important, value of the study of geometry in the high school lies in the fact that it leads the pupil into independent thinking and possibly to logical reasoning. It certainly is not an aid to this for a teacher to believe his Euclidean geometry much like he believes his Bible and to give the impression to the pupil that these axioms have the same nature as the Ten Commandments (minus the religious value) and that the theorems set down in the book and proved there must be learned verbatim. Influenced by such an attitude on the part of his teacher, the pupil becomes a slave to authority and seldom, if ever, thinks to question the validity of any statement seen in a book. The pupil's attitude toward mathematics and mathematical thinking should be properly nurtured if he is to have the proper conception of mathematics. It can not be so nurtured by a teacher who does not have the proper conception himself. The pupil of the ninth or tenth grade can not be taught non-Euclidean geometry, indeed may not necessarily hear the name, but he can be made to realize that the conclusions of geometry depend on arbitrary (though reasonable) assumptions and hypotheses and that the changing of a postulate would produce different theorems. He can be led to speculate on what would happen if . . . . . He can realize that the conclusions in geometry as in life are not all set before you begin, but depend on the premises at the outset and conditions that may be injected later. What it means to a teacher to know about non-Euclidean geometry is illustrated by the following true story.

When I was a young man I set out on horseback one spring morning with a Missouri Teacher's Certificate in my inside pocket and an older brother who had had several years' experience teaching country schools riding by my side. The object of the journey was to meet the Board of Directors of a rural school a few miles away and secure, if possible, a contract to teach there during the coming term.

When we had finally assembled the members of the Board at the home of the Chairman, my brother led in the conversation, as he had had some experience in "getting schools" while I had had none. My certificate was produced and he guided the inspection thereof, pointing with pride to the grades made in the teachers' examination as they were recorded on the certificate. They ran like this: Arithmetic, 95; Grammar, 100; etc., etc., . . . Pedagogy, 68! Here was an unfortunate situation which seemed to demand defense and explanation.

My brother began, "Well, you see he has not had any experience yet and naturally couldn't answer questions as well in that subject—"

"Well", broke in the Chairman, "he won't have to teach nothin' like that here, nohow."

Too many educational administrators to-day by their methods of selecting teachers of mathematics are unconsciously betraying the fact that their conceptions of a teacher's need of subject matter above what he teaches is identical with that of this unlearned farmer toward a teacher's need of pedagogy.

It seems to me that it is as important for a teacher of geometry (or any other subject) to know many things about the subject and its relation to mathematics that he can not teach his pupils as it is to know many educational theories not all of which he uses.

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PART TWO

METHODS OF TEACHING MATHEMATICS IN THE JUNIOR AND SENIOR HIGH SCHOOLS

CHAPTER VI.

GENERAL DISCUSSION OF METHODS

55. **Introduction.** In the discussion of the teaching of mathematics in this and the remaining chapters let it be understood that only basic suggestions are to be made, suggestions of methods based on the true nature of the subject as discussed in the preceding chapters. More detailed discussion of the methods of presenting algebra and geometry may be found in the books on the teaching of mathematics mentioned elsewhere herein, and in the mathematical journals. Better results can be obtained by a teacher who knows the subject and who knows well and practices a few tried and proved methods than by a teacher who knows something about a lot of theories of teaching which he does not understand well enough to use intelligently. **The best methods for a teacher are those evolved by the teacher himself** after much practice, provided they are sound mathematically and provided they get results. A beginner, however, must have some borrowed methods to make a start. These chapters are written in order to help the beginner.

I. METHODS PERTAINING TO THE SUBJECT MATTER

56. **The Synthetic and the Analytic Methods.** The synthetic method refers to the method of proceeding by logical steps from a hypothesis to a conclusion, from the known to the unknown. No clue is given as to the reason why each step is made; it is only made. It is the method of presentation of finished logical proof as exemplified by the usual proofs of the propositions in geometry.

The analytic method is a kind of reasoning which proceeds from the desired conclusion, step by step, until contact is made with the hypothesis, that is, reasoning from the unknown to the known. Before being selected, each step is tested and approved as to its possibility of being a link in the chain of proof. The analytic method is the method of discovery of a new proof.

Two couriers desire to cross a wide area under the eyes of sharpshooters in order to reach a desired goal on the other side. The first selects a cover a short distance away and creeps to it. Reconnoitering under its protection, he selects another further on and creeps to it. Pursuing this policy, he continues and, if his sense of direction is good, will arrive at his destination in due time. At each point of his progress he was sure of no more than one successful step ahead. The second courier surveys the goal with his field-glasses. He sees a short distance back from it a shelter from which, if he could reach it, he might complete the journey. Back from this he "spots" another from which he could reach the first unseen. Continuing in the selection of a possible route (in reverse direction) he finally selects a cover that he can reach from where he stands. His course is mapped out. He is ready to begin his journey with certainty of success, knowing the location of every stopping place on the way. The first courier's method of progress is analogous to the synthetic method, the second to the analytic.

As an illustration consider the following theorem:

The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.

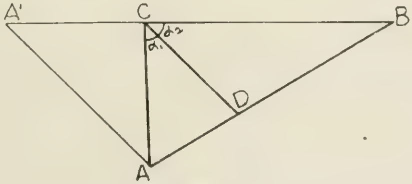


FIGURE 13

Given: The bisector of the angle C of the triangle ABC meeting the side AB at D.

To Prove: $\dfrac{AC}{CB} = \dfrac{AD}{DB}.$

Synthetic proof: (The reasons for the steps are omitted.) Extend BC making $CA' = CA$ and draw AA'.

$\angle A' = \angle CAA'$.

$\angle ACB = \angle A' + \angle CAA' = 2\angle A'$.

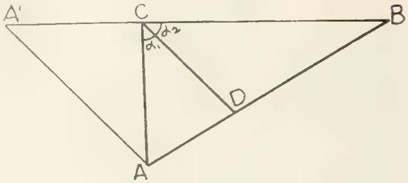
$\angle \alpha_1 = \angle \alpha_2 = \frac{1}{2}\angle ACB$.

$\therefore \alpha_2 = \angle A'$.

$\therefore AA'$ is parallel to CD.

$$\therefore \frac{A'C}{CB} = \frac{AD}{DB}.$$

$$\therefore \frac{AC}{CB} = \frac{AD}{DB}.$$



Analytic proof: A proportion like the one to be proved can be found if the four segments involved are the segments formed on two sides of a triangle by a line parallel to the third side. AD. DB and CB are in correct position for such a proportion. If CA were an extension of BC and if a triangle could be formed so that CD is parallel to the side opposite the angle B, the proportion would be true. Hence we extend BC to A', making CA' = CA, and draw the line AA'.

$$\frac{AC}{CB} = \frac{AD}{DB} \text{ if } \frac{A'C}{CB} = \frac{AD}{DB}.$$

$$\frac{A'C}{CB} = \frac{AD}{DB} \text{ if } AA' \text{ is parallel to CD.}$$

AA' is parallel to CD if $\angle \alpha_2 = \angle A'$.

Since $\angle \alpha_2 = \frac{1}{2}\angle ACB$ this would be true if the triangle ACA' is isosceles.

The triangle ACA' is isosceles if CA' is made equal to CA.

Hence we begin by making CA' equal to CA and reverse the order of the steps in the analysis which gives the synthetic proof just preceding.

In the synthetic proof the first step is made without any suggestion whatever as to the purpose or the subsequent plan of proof. It would probably never be thought of unless preceded by an analysis like that given in the analytic proof.

As another example, take a familiar theorem of algebra:

If four numbers are in proportion they are in proportion by addition (composition).

Given: $\frac{a}{b} = \frac{c}{d}$.

To prove: $\frac{a+b}{b} = \frac{c+d}{d}$.

Synethic proof:

$$\frac{a}{b} + 1 = \frac{c}{d} + 1.$$

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

Analytic proof:

$$\frac{a+b}{b} = \frac{c+d}{d} \text{ if } \frac{a}{b} + 1 = \frac{c}{d} + 1.$$

$$\frac{a}{b} + 1 = \frac{c}{d} + 1 \text{ if } \frac{a}{b} = \frac{c}{d}.$$

which is given as the hypothesis. Now by reversing the steps in the analysis we obtain the synthetic proof.

In the synthetic proof the addition of 1 to both sides of the equation is like a step in the dark to one who has not previously analyzed the theorem.

The conventional method of solution of an algebraic equation is analytic. For example, solve

$$2(5x + 1) + 16 = 4 + 3(x - 7). \qquad (1)$$

We shall know the solution of (1) if we can find the solution of

$$10x + 2 + 16 = 4 + 3x - 21. \qquad (2)$$

The solution of (2) is determined when we find the solution of

$$10x + 18 = 3x - 17. \qquad (3)$$

We shall know the solution of (3) if we can find the solution of

$$7x = -35, \qquad (4)$$

which is

$$x = -5.$$

To make the synthesis (which is of no value) we should have given

$$x = -5.$$

Then

$$7x = -35, \text{ (Equals multiplied by equals)}$$

$$10x + 18 = 3x - 17, \text{ (Equals added to equals)}$$

and so on to the end. The synthetic method is not used in algebra.

To adhere to the synthetic method, or to merely follow the synthetic proofs as given in the text in geometry, is to promote the bad habit that is generally recognized and unanimously opposed by good teachers; namely, the habit of memorizing proofs. It taxes the ingenuity of the very best of teachers to make all the steps in the usual synthetic proofs seem so reasonable to the pupil that he might be expected to reproduce them by a course of independent thought rather than by remembering what comes next. A judicious use of the analytic method will be of great value to the teacher. This method of dealing with the proved propositions is discussed more at length in Chapter XV.

Shall we then abolish the synthetic proof? By no means. It remains the clear, simple, concise manner of recording proof and pupils should become acquainted with it.

57. **The Deductive and the Inductive Methods.** By deduction is meant the process by which a particular truth is derived from some general known truths. A perfect example is an ordinary syllogism like the following:

All chickens have feathers.
This is a chicken.
Therefore it has feathers.

The finished science of mathematics is a deductive science[48]. It abounds in syllogisms like the following:

(1) If in a quadratic equation of the form $ax^2 + bx + c = 0$ we find that $b^2 - 4ac < 0$, the roots are imaginary.

In the equation $2x^2 - 4x + 7 = 0$ it is true that $(-4)^2 - 4 \cdot 2 \cdot 7 < 0$.

Therefore the roots of $2x^2 - 4x + 7 = 0$ are imaginary.

(2) All right angles are equal.
These two angles are right angles.
Therefore they are equal.

A synthetic proof is a series of deductions.

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[48] See the first definition by Bertrand Russell quoted in §6

By induction we mean that process of assuming a general truth as the result of having observed its verification in many particular cases. In this case the procedure is from the particular to the general in contrast with the procedure from the general to the particular in the deductive method. Induction does not prove; it discovers. It finds the material for the deductive method to prove. Mendel's Law of inheritance, Boyle's Law for gases, and empirical formulas of practical mathematics are all discovered by inductive methods. Let us consider two illustrations from mathematics, one where the inductive method has no place and one where it may be used.

Suppose we are teaching the theorem in plane geometry about the sum of the interior angles of a triangle. The pupils may measure the angles in many triangles and average the sums, but it in no wise establishes the fact that the sum is actually 180°. The exercise bears no relation whatever to the proof that we are about to make of the theorem.[49]

Suppose instead that the theorem about the sum of the angles of the triangle has been established. The pupil is encouraged to investigate the sums of the interior angles of other convex polygons, the quadrilateral, the pentagon, etc. He divides each of them into triangles by means of diagonals drawn from the vertex of one angle. A tabulation of his observations would yield the following facts:

Number of sides of the polygon	3	4	5	6	7	8
Number of Δs into which it is divided	1	2	3	4	5	6
Sum of the ∠s (expressed in rt. ∠s)	2	4	6	8	10	12

A study of this table reveals the fact that the number of triangles (as far as he has observed) is always two less than the number of sides and the sum of the interior angles is twice as many right angles as the number of triangles. If the polygon has n sides this would mean that the sum of the interior angles is $2(n-2)$ right angles. No amount of trial can ever establish this as a truth satisfactory to mathematics for we can never try all possible cases. The inductive method has suggested, however, a method of proof while suggesting the possible truth itself. Using this method, we establish the general formula: sum $= 2(n-2)$ rt. $\angle$s.

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[49] This exercise would not necessarily be taboo in the intuitional geometry of the seventh and eighth grades. We do not claim that there may not be inductive methods applied to this theorem which will be illuminating, but this particular one in this place (formal geometry) is not good.

We find the inductive method illustrated in practically all texts in algebra by a series of exercises much like the following:

$$\begin{array}{rrrr}
x + 3 & x - 3 & x - 3 & x + 3 \\
x + 5 & x + 5 & x - 5 & x - 5 \\
\hline
x^2 + 3x \phantom{{}+15} & x^2 - 3x \phantom{{}-15} & x^2 - 3x \phantom{{}+15} & x^2 + 3x \phantom{{}-15} \\
5x + 15 & 5x - 15 & -5x + 15 & -5x - 15 \\
\hline
x^2 + 8x + 15 & x^2 + 2x - 15 & x^2 - 8x + 15 & x^2 - 2x - 15
\end{array}$$

In each case attention is called to the fact that the coefficient of the common term x in the product is the algebraic sum of the two constant terms of the factors and the constant term in the product is the product of the two constant terms in the factors. This leads us to surmise the general formula and prepares our minds for its form; namely, $(x + a)(x + b) = x^2 + (a + b)x + ab$, which we proceed to establish. The use of induction brings a pupil gradually to the general formula which if given at the outset would be hard to understand.

When we have established the general formula, we infer by deduction that if we multiply $(x - 4)(x + 6)$ we will obtain $x^2 + 2x - 24$ and need not take the time to multiply.

58. Conclusions. The use of the analytic and inductive methods in teaching mathematics will serve the great purpose of developing a spirit of inquiry (some may want to call it research) in the student. His ingenuity must be drawn upon and habits of independence and originality of thought will be the result. Induction leads him through particular situations, concrete to him, to an appreciation of a general truth whose very generality if placed before him at the beginning would probably place it beyond his comprehension.

The synthetic and deductive methods serve also the aims of education. Nowhere in the whole scale of learning processes and nowhere in the whole list of high school subjects is there such excellent opportunity to learn simple, accurate, concise and powerful modes of expression as in the statement of a synthetic, deductive proof. The synthetic and deductive methods should follow the analytic and inductive methods. If the synthetic and deductive methods come first the process of learning will probably degenerate into a memorizing process.

Let us, by a judicious use of the inductive and analytic methods, inspire in our pupils the spirit of the intellectual pioneers of the past who, by force of their ingenuity, resourcefulness, and originality, pushed out into the intellectual unknown to find the truths that awaited them there.

II. STANDARD METHODS PERTAINING TO THE MANNER OF PRESENTATION

59. The Lecture Method. This method, adapted to adult students of responsibility and especially useful in large classes of such students, has no place in the teaching of high school mathematics. It is mentioned here for the sole purpose of pointing out a defect in our teaching which most of us have; namely, telling too much without asking questions and getting responses from the students. Not one of us teaches mathematics by lectures as the university professor lectures to an advanced class, but we lecture, nevertheless!

Is it meant that we can not explain parts of a lesson without being guilty of lecturing? Not unless our explanations are strictly in answer to the pupil's question and confined to answering the question. When a teacher gives voluntary information upon a part of the lesson that can be learned as well by the pupil through his own initiative under proper guidance and encouragement he is using the lecture method. He is also reducing the pupil's chances for independent thinking. Everybody knows that the impression made in a pupil's mind by the discovery of a fact himself is of much greater value than the impression made by being told the fact by the teacher. The main excuse for resorting to the lecture method is that much more can be accomplished in the brief time allowed for the recitation period. As long as we have large classes we will find our recitation moving more slowly than we anticipate and in order to finish the lesson we sometimes dispense with our thought-developing questions and assume (probably unconsciously) the role of lecturer. Those occasions should certainly be few and far between.

60. The Heuristic Method. Diametrically opposed to the lecture method is the heuristic method (from the Greek, "euriskein", to discover). It is the process of leading the pupil by skillful questions to find the desired knowledge himself. It involves very little telling of facts. Consider this illustration of the method as applied to an individual pupil seeking help on a problem in algebra.

The problem: A man has $8.00 in nickels, dimes, quarters and half dollars. He has 3 times as many nickels as quarters, twice as many half dollars as quarters and half as many dimes as he has nickels and quarters together. How many of each kind of coin has he?

Teacher: What is given? (The pupil states the numerical facts concerning the amount of money and the relations between the numbers of the various kinds of coins.)

Teacher: What is to be found?

Pupil: The number of each kind of coin.

T. How many kinds of coins has he?

P. Four.

T. Which number to be found will you choose to be represented by the symbol for the "unknown"?

P. I don't know.

T. Is there one whose value depends on another?

P. I don't know.

T. Read the first part of the second sentence and try to answer the question.

P. (After reading) Yes, when we find how many quarters there are we can know the number of nickels.

T. Read on and tell me if you find other dependent numbers.

P. Yes.

T. Is there any kind of coin whose value is not stated as depending on some other?

P. Yes, quarters.

T. How many of the other unknowns can you get if you know the number of quarters?

P. All of them.

T. If you choose the number of quarters as your "unknown", can you express the others in terms of that?

P. Yes, if q = the number of quarters,
$3q$ = the number of nickels,
$2q$ = the number of half dollars, and
$2q$ = the number of dimes.

T. Can you form an equation?

P. Yes, $q + 3q + 2q + 2q = 8.00$.

T. Be careful! What does q represent?

P. The number of quarters.

T. Suppose it is four quarters. If q represents the **number** of quarters does $q = 4$, or does $q =$ one dollar?

P. $q = 4$.

T. Does $q + 3q + 2q + 2q$ represent "how many **coins**", or "how many **dollars**"?

P. It represents how many coins.

T. Does 8 represent how many coins?

P. No.

T. Can you say, "The number of coins is equal to eight dollars"?

P. No.

T. What do you need in the first member of your equation to replace the number of coins if the second member is expressed in dollars or cents?

P. The value of the coins.

T. Very well; what is the value of q quarters?

P. I do not know. What is it?

T. What is the value of one quarter?

P. Twenty-five cents.

T. What is the value of two quarters?

P. Fifty cents.

T. What is the value of three quarters?

P. Seventy-five cents.

T. How did you get the last answer?

P. By multiplying twenty-five by three.

T. How then would you get the value of q quarters?

P. By multiplying twenty-five by q.

The remainder of the dialogue can easily be supplied by the reader.

Consider now an illustration of the method as it might be used with a whole class in geometry. Suppose the teacher is developing with the class as an original theorem, before assignment as a lesson, the proposition: In an isosceles triangle the angles opposite the equal sides are equal. The answers are not all supposed to be given by the same pupil, but by different pupils as indicated by the teacher in the course of the development.

Teacher: (After drawing on the board all of Fig. 14 except the line CD). State the hypothesis, using the notation on the figure.

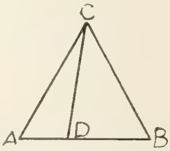

FIGURE 14

Pupil: AC = CB.

T. What is to be proved?

P. ∠A = ∠B.

T. What methods of comparing equality of angles have been established?

P. Vertical angles are equal.

P. Right angles are equal.

P. Straight angles are equal.

P. Corresponding angles in congruent triangles are equal.

T. According to the position or nature of the angles A and B which of these conditions do they fulfill?

P. None.

T. Can they be made to fulfill any of the conditions?

P. If we draw a line from C intersecting the base AB the angles A and B will be in two different triangles.

T. Are the triangles thus formed congruent?

P. That depends on how we draw the line through C.

T. Suppose we draw a line. How many corresponding parts in triangles ADC and BDC must be equal?

P. Three.

T. How chosen?

P. Either two angles and the included side or two sides and the included angle.

T. How many parts are given equal by the hypothesis?

P. One side in each, AC = BC.

T. Do you see any other part or parts equal in the figure?

P. CD belongs to both triangles.

T. Good! That makes two sides; what next?

P. The angle ACD must be equal to the angle BCD.

T. Is it?

P. That depends on how you draw the line CD.

T. How did we draw CD?

P. Any way to make two triangles.

T. How shall we draw it to make the triangles congruent?

P. So that it bisects the angle ACB.

T. Very well. As our first step let us draw the line CD so as to bisect the angle ACB. What statements can you make on which to build a proof?

P. AC = CB, CD = CD, ∠ACD = ∠DCB.

The remainder of the development is evident.

It is the aim of the pure heuristic method to tell a pupil nothing that he can discover himself. If that ideal can be maintained, the method transcends all others; but so long as a teacher has thirty or forty pupils in a class, and possibly six or seven such classes every day, there is no possibility of using a pure heuristic method, for it requires time and attention to the individual. If in class teaching the teacher calls for an answer as soon as a few pupils show readiness to answer then it becomes a heuristic method applied only to a few and a perverted form of the lecture method for the slow ones where the alert pupils who give the answers play the roles of tellers, or lecturers, instead of the teacher. If the teacher does not call for the answer until all (or nearly all) of the pupils indicate readiness, very little can be accomplished during the recitation period. The teacher must wisely control the situation as as to allow as much independent thought by all pupils as possible and at the same time not move too slowly. It takes a wise teacher to strike a proper balance between telling and asking. Fortunate is the teacher that has a small class. We may say, fortunate is the community that supports its schools in such a manner that the teachers may have only small classes.

It should be mentioned that even a slow pupil gets more benefit out of the act of trying for a brief time to answer a question, even though the answer is given before he has formulated one, than if the fact were given him directly without asking for productive mental effort on his part.

The heuristic method sometimes degenerates into a form of questioning that is worthless, that is, when the questions suggest the answers. A pupil may as well be told that two angles are equal or that one line is parallel to another as to be asked, "Are these two angles equal?", or, "Is this line parallel to that?"

61. The Recitation Method. Well do I remember the kind of teaching received in an ungraded elementary school during the final decade of the nineteenth century. I shall take as a type of all the class periods the one in geography. The vision of the textbook page is as clear as if I saw it only yesterday. Each short section of descriptive matter was followed by several questions of fact about it. We pupils endeavored to memorize the subject matter. When the class assembled on the long recitation bench at the front of the schoolroom the teacher, with open book, said, "Close your books", and we did. The teacher read the questions on the sections included in the lesson and one by one we answered them, or failed to answer. The next lesson was assigned and we returned to our seats. This is a pure type of the recitation method. The object is to find out what the pupil has learned (memorized?). No teaching is done. No thought is stirred up. The pupil recites and the teacher marks the grade. Memory is tested and that is all.

It must not be assumed that the teacher should never use the recitation method. There are situations arising all the time wherein the teacher will need to get answers to direct questions of fact in order to test the results of his teaching. The recitation method is not bad unless used to the exclusion of other methods.

62. **The Laboratory Method.** In 1901 Professor John Perry gave an address on the teaching of mathematics before the British Association at Glasgow in which he advocated less emphasis on the formal and more on the practical, or utilitarian, side of mathematics. He urged a closer relation between the method of teaching and the applications of mathematics in the other sciences, suggesting experimental work by the pupil. The idea was given great impetus in America by the Presidential Address of Professor E. H. Moore before the American Mathematical Society in 1902.[50] Professor Moore said, in part: "As a pure mathematician I hold as a most important suggestion of the English movement, the suggestion of Perry's just cited, that by emphasizing steadily the practical side of mathematics, that is, arithmetic computations, mechanical drawing, and graphical methods generally, in continuous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by laboratory process, and, on the other hand, by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics".

The outgrowth of the "Perry Movement" is a method of teaching which may be called the laboratory method. Just what it involves depends on the particular advocate of the method. Some would have areas, volumes, and other formulas of mensuration determined by experiment. Some would consider as a laboratory exercise the step by step method of leading up to a general rule through special to more general cases, that is, a process of induction, using the symbols of mathematics rather than material laboratory equipment. Some would make the course in geometry principally a course in mechanical drawing and the course in algebra a course in graph plotting, approximate computation and experiments with balances, pendulums, pulleys, thermometers, etc. A wise teacher may make judicious choices among all these types of exercises and use the method in a way to enrich, enliven, and improve his teaching, bringing the pupil face to face with practical applications and leading him by a concrete inductive exercise to appreciate a formal proof. An injudicious and inexperienced teacher can easily ruin the course in mathematics.

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[50] See *Bulletin of the American Mathematical Society*, 1903, 402-424.

Some experimental work is worth while; some is not. What does or does not constitute a worth while laboratory experiment depends on the viewpoint of the teacher. Probably no one person should be the judge. I submit the following criteria by which one may test the worth of a laboratory exercise:

(1) Does it actually make clear to the pupil the possibility of practical application of his mathematics? (2) Does it illuminate or suggest a method of mathematical proof or reasoning?

Some exercises do neither, but merely take up a student's time with experimentation. For example, suppose the area of a trapezoid is experimentally determined by drawing several on squared paper and counting the squares contained. The relation of the number of squares to the lengths of the bases and the altitude would probably never be discovered by such an exercise. If the formula is given at the beginning, then the exercise becomes no more than a childish verification. An exercise might be devised which involves the cutting of a paper trapezoid into triangles which would lead to a discovery of the area formula and suggest a method of proving it. Such an exercise would for that reason have real value.

Let us consider two possible laboratory exercises in algebra, one satisfying each of the criteria set up. (1) In the study of the formula (or the equation) in algebra the law of the lever is considered. Two boys, weighing x pounds and y pounds, respectively, can take some weight, say a twelve-pound shot, and by means of a "teeter" board carefully balanced can determine their weights. For, if a, b, c, and d (considered as measured distances) represent the lever lengths, the following equations are obvious. $a(x + 12) = by$; $cx = d(y + 12)$. (2) The familiar geometric figures illustrating the formula $(a + b)^2$

and $(a - b)^2$ can be quickly constructed on squared paper, the pupil making several of them, using various values for a and b. The formula can be developed by the inductive method, using this laboratory exercise.

The illustration (§57) of the inductive method as applied to the theorem on the sum of the interior angles of a convex polygon suggests a type of laboratory exercise without instruments. As an example of a laboratory exercise that by its nature reveals the successive steps of a formal demonstration we suggest the following exercise in paper folding.

Let the pupil take a rectangular piece of paper and fold so that the crease (a straight line) will intersect one side of the paper, making oblique angles with it. In Fig. 15 CA represents the straight side of the paper and OB the crease. Now if OA and OC are folded over on OB the creases made will be bisectors as is obvious by the exact coincidence of angle AOP with POB and COQ with QOB in the folded figure. Also, the folded figure shows that the angle POQ is half of the straight angle AOC. Thus the pupil discovers that the bisectors of two supplementary angles are perpendicular to each other. He has also performed with the folded paper, analogous steps to those he must take in making a formal proof.

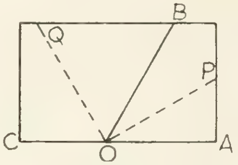

FIGURE 15

An ingenious teacher can devise paper folding exercises[51] to illustrate many other theorems. Such exercises require no expensive equipment nor special room.

References

Young, The Teaching of Mathematics, Chap. IV.
Schultze, The Teaching of Mathematics, Chap. III.

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[51] See Row, *Geometric Exercises in Paper Folding,* Open Court Publishing Company, Chicago.

CHAPTER VII

PROJECT TEACHING

63. Introduction. In recent years the new name "project" has arisen on the educational horizon and to some it heralds the light of a new day of educational achievement. In some places it is a name to conjure with. There are those who naturally tend to magnify any method that is modern who have let the "project" idea completely dominate their educational procedure. Because of its far-reaching influence in educational circles to-day, a whole chapter will be devoted to a discussion of project teaching.

64. Definition. Each independent leader of educational thought has his own definition of the term, project. Stevenson[52] quotes eight definitions, using the names of twelve leading educators, and gives his own definition as follows: "A project is problematic act carried to completion in its natural setting". Laying aside the definitions which he quotes from men particularly interested in special phases of education, there are five definitions "proposed by teachers whose field of activity is that of general educational theory"; namely, Charters, Kilpatrick, Babbitt, Stone and Snedden. No two of these exactly agree as to the meaning of the term. Some, like Stevenson, insist that there be a "natural setting" for the project. Others agree that it should be "carried to completion". Some stress "concrete achievements", or the manual side of the process. One common factor in all the viewpoints is that the project is an act of the pupil, by the pupil, and for the pupil. In other words, it is not a process wherein the pupil passively submits to a teacher's prearranged program of drill or topical memorization planned for him to carry out, but rather a more or less independent piece of work performed by the pupil himself because of its appeal to his major interests.

Kilpatrick, writing after the publication of Stevenson's book cited above, defines project as follows[53]:

> The particular word, project, is of small consequence; the idea or point of view back of the word is the important element. We understand the word project to refer to any particular unit of purposeful experience, any instance of purposeful activity where the dominating purpose, as an inner urge, (1) fixes the aim of the action, (2) guides its process, and (3) furnishes its drive, its inner motivation.

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[52] *The Project Method of Teaching,* Chap. III.

[53] *Teachers College Record,* Sept. 1921, pp. 283-288.

The project thus may refer to any kind or variety of life experience which is in fact actuated by a dominating purpose. I myself distinguish four types which in their border cases merge, to be sure, into each other. Moreover an example of any type might conceivably appear as a subordinate purpose under any other of the four types or under another instance of its own type. Let us consider the four types in turn.

The first type represents those experiences in which the dominating purpose is to do, to make, or to effect; to embody an idea or aspiration in material form. The material of which the thing is made, in which the idea is to be embodied, may vary from clay, wood, cloth, and the like, through marble or pigment, to the words and thoughts and aspirations of a letter, a speech, a poem, a symphony, and a prayer. The finished production may be as insignificant as a child's sand pile on the seashore or as great as Alexander's empire. It may be built in a moment and perish in the building, or it may take as long in the making and remain as enduring as Newton's *Principia*. The criterion for judging is the character of the purpose. Is there an idea to be embodied? Is there an animating purpose to realize the idea? Is there consequent effortful action dominated by this purpose? If 'yes' is the answer to all these questions, then the project is of the first type.

The second type of project may be defined as one which involves purposeful enjoying or appropriation of an experience. A boy will see and enjoy fireworks, or a circus, or a parade of soldiers. . . . . . His mother will with a similar true purpose see and enjoy a sunset, or the beauties of the Mona Lisa. . . . . Experiences of this type are, in comparison with those of the first type, relatively passive. . . . . If the experience were in fact an entirely passive one, then purpose would have no place. But in all such experiences there is much activity. So again we ask: Is there a purpose for engaging in, and appropriating, and enjoying the experience? Does the purpose guide the action of seeing or hearing, as the case may be? If there is this purpose, then the experience described is a project of the second type.

The third kind of a project is one in which the dominating purpose is to solve a problem, to unravel and so compose some intellectual entanglement or difficulty. The problem has its natural setting and origin, at least in race history, in the pursuit of some end. Thus it begins, both for the individual and for the race, as a subordinate part of a project of the first type. Probably for most people thinking is limited largely to such practical situations as arise in ordinary life: a difficulty arises; thinking is necessary to surmount it. . . . The criterion is as elsewhere, the presence or absence of a dominating purpose. . . . . A project of the third type implies first a felt difficulty, a problem; and second, a purpose to solve the problem. The use of problems being granted, the part that purpose plays in solving them, especially, the more complex ones, is so clear and definite that none will question the proper inclusion of this as a third type of project.

The fourth type includes experiences in which the purpose is to acquire some item or degree of knowledge or skill, or more generally, experiences in which a person purposes his own education at a specific point. The difference between this and other kinds of drill is again exactly one of attitude. Here the child purposes to learn the thing at hand, an attitude which makes a great difference in the efficiency of learning. A particularly valuable purpose in the realm of school work is one in which the person purposes to organize a point of view already more or less in hand, and to fix it in his memory for effective use later. . . . . .

We may further distinguish group projects from individual projects In

> the latter one person alone is considered as feeling the dominating purpose. In the former several unite in a common purpose and pursue cooperatively, by a more or less clearly marked division of labor, the end held jointly in view. The social value of such cooperative pursuit of joint purposes needs no discussion here.

Professor Kilpatrick goes on to make in succession the points, (1) that it is not necessary that the child suggest the project, but that either the child or the teacher may do so, provided the child so feels the purpose of the plan "that it operates as an inner urge to define the end, guide the pursuit, and supply the drive;" (2) that "a project is not a topic"; (3) that methods of using projects differ widely, no cut and dried curriculum being possible, much depending on the originality and ingenuity of the teacher who must decide which, if any, of the pupils' offerings should be utilized to best serve the aims of education. He then concludes with the following paragraph:

> We thus face the crux of the problem immediately at hand: on the one hand the project method is committed to wholehearted child purposing as fully as it may be got; on the other, society and ethics unite in demanding that the child increasingly incarnate the assured values of race experience. Is there any contradiction between these two demands? What difficulties and dangers does the teacher face in the practical effort to meet both? We do grant, and we must grant the demands of society and ethics. In view of this what dangers and difficulties arise and how shall we overcome them, if we are to conduct our schools on the project basis? In endeavoring to solve this problem we must bear in mind the fundamental fact, that a project is any unit of experience dominated by such a purpose as sets an aim for the experience, guides its process, and furnishes the drive for its vigorous prosecution.

65. Project Method in Teaching Mathematics. Every time a pupil becomes so interested in finding the result in a verbal problem in algebra or arithmetic, or an exercise in geometry, that he has an "inner urge" which impels him to prosecute the solution to a successful conclusion, he engages in a project of the third type mentioned by Kilpatrick in the quotation above. If a pupil possesses a desire to learn the multiplication table, or the methods of factoring polynomials, or a method of solving quadratic equations, to such an extent that he is furnished by means of that desire with motive power to drive on to success, he engages in a project of the fourth type. The use of projects of these two types by teachers of mathematics antedates by many centuries the consideration of the project method in educational theory. The proponents of the project method do not claim any copyright on the term. Some of them insist that there is little that is new in educational theory in project teaching. It seems to be mainly a method of placing the emphasis in the educational process on securing independent activity on the part of the pupil, thereby encouraging the development of initiative and originality. If this be a true interpretation of project teaching then all good teachers of mathematics can give hearty support to the general idea, at least.

The problem for the classroom teacher is to so direct the activities of the class that the pupils may feel the "inner urge" to solve the problem, prove the original exercise, learn the multiplication table, or the methods of factoring, or of solving quadratics. In other words it is the age-old problem of motivation. According to legend, Euclid met the problem when one of his royal pupils demanded a reason for studying geometry. Euclid failed to rise to the educational possibility of the situation and use the project method. He preferred, according to the story, to ridicule the prince by ordering his slave to reward him with some small change if his pupil must see gain in his study. Euclid could see no need for motivation other than the love of the subject.

In the attempt to solve this problem of motivation textbook writers have made verbal problems relate to the customary activities of child life and to the everyday experience of the average person in earning a living. Others have appealed to the pupils' curiosity and love of the mysterious by means of "puzzle" problems. Teachers have exploited the very natural human desire to excel by using contests to motivate the acquiring of skill in mechanical processes. Some have found the point of contact with the concrete through laboratory exercises. In many cases a laboratory exercise is a project. However, when the laboratory exercise takes on the aspect of an assigned task to be done by the pupil who has no desire to do it except to satisfy his teacher that he may get a grade it is not a project.

It will tax the ingenuity of a teacher of mathematics to so inspire his pupils that he may get from them either the suggestions for projects of the right kind at the right time (for a course in mathematics must be developed in an orderly, and in some cases logical, manner) or a wholehearted enthusiasm in engaging in the projects suggested by the teacher. A teacher that can do this successfully will usually succeed by using any kind of reasonable methods. The poor, or mediocre, teacher in all probability lacks the particular ability to give the class the proper spirit to carry on by means of project teaching. Such a teacher will perhaps be a poor, or mediocre, teacher whatever method he uses. It seems that the method should be judged not so much on its convenience from the teacher's standpoint as from the results secured, or, in other words, its effect on the pupils. No teacher of mathematics who has labored for years trying to develop habits of independent thought, careful consideration of data given in any problem, and the exercise of initiative and ingenuity in reaching a conclusion will fail to see in the use of project teaching a chance to encourage the growth of those habits and virtues he has so long been trying to instill into the lives of his pupils.

66. Misconceptions. Some subjects in the curriculum of the high school do not provide in the subject matter contained the opportunities for independent activity that is provided in problem solving in mathematics. Many teachers of those subjects, and a number of teachers of mathematics, have so conducted their work in the past that the pupils learned by passive assimilation, or by sheer memorization, rather than by the active, independent exercise of their own powers of thinking. In reacting against this some educational administrators have seized upon project teaching as a panacea for all educational ills and have made it control all teaching in the schools under their supervision. School-wide schemes, called projects, have been devised which, because of their bizarre nature and impressive novelty, arouse much interest and activity among the pupils. A "World's Fair Project" dominates the activities of a certain grade school for five or six weeks. Each grade has its own project to work out and the list includes Indians, Holland, Circus, China, Japan, Transportation, Patriotism, South America, Europe, Birds, and Art Gallery. Reading, arithmetic, penmanship, spelling, language, etc., each has its place during this period only as it occurs incidental to the main program. But, suppose a fractional computation is needed in the course of the development of the project before that particular class has learned multiplication. Or, if the development of the project is so engineered by the teacher that all arithmetic which occurs will come in its proper order, what will be the effect on the order of subject matter in the other subjects? Can the words arising which must be spelled be properly graded? Of course it is not impossible to organize the project so that the individual subjects may be presented in a more or less orderly manner, but in so far as it is artificially molded by the teacher to fit certain conditions it fails to be a pure project suggested by the group or by individuals in the group. Would it not be better to administer project teaching in small "doses," allowing each project to grow naturally out of the subject matter of the course?

In one high school where project teaching is in vogue some of the pupils have betrayed the fact that when asked to suggest projects they always vote to take some trip in order to get away from school. In the same school some members of a class in eleventh grade English which was supposed to be studying American literature spent several recitation periods wandering about the school grounds gathering material to construct a model Indian village. In order that this class might get inspiration for projects a fifth grade book of stories about American colonial history was used as a text for a few weeks. Some of the larger boys of the class became disgusted when one of their number happened to read in the preface that the book was designed for the use of pupils from ten to twelve years old! Many of the class at the end of the semester chose other subjects rather than continue the course.

Surely we must not, because of our devotion to a method, however sound its fundamental principles may be, close our eyes to certain common-sense facts about young people. They do not always remain "kindergarteners", though the easily excitable enthusiasm about some little thing that can be engendered in some of them is very discernible. Some of them have learned enough since childhood days to take advantage of a teacher's hobby to sidetrack education for the prosecution of schemes for their own pleasure.

Neither must we forget to preserve in our teaching "the assured values of race experience". There is educational value in the study of colonial literature. Should we lose precious time from the study of the literature itself to turn aside in some project suggested by a class which may only in part relate to the subject at hand; namely, building an Indian village? The author of this bulletin remembers well his first study of American literature when a lad seventeen years old. He would have welcomed the opportunity to engage in the propect of producing some original verse, the criticism of which would have taught him much about poetry. Such a project, which is often used, illustrates what we mean by one which grows out of the subject matter of the course.

Though we have at hand no first hand information covering the matter, it is easy to see that in mathematics a teacher might go astray if he persists in always using big comprehensive projects. It is hard to believe that the students of the class in algebra can decide what they should undertake next further than one or two lessons.

67. Conclusions. It is my opinion that in no case should the project subordinate the subject but should be subordinate to the subject. Especially should it be so in mathematics where the nature of the subject is such that within it are found two of the four types of projects mentioned earlier in this chapter. If a teacher can get a suggestion from the pupils that they would like to try the proportionality theorems about similar triangles in measuring the height of the flag pole, he has a good project. Or is it not merely a laboratory exercise? It is but a sample of many schemes that may be used to vitalize the teaching of mathematics, schemes that were used by a multitude of teachers before the word project invaded educational literature. It is not the aim of this bulletin to take the place of a manual in suggesting a list of possible projects. The pupils can not catch the spirit of independent thinking and originality from a teacher who must borrow all his plans from others. The teacher who does not possess the initiative and ingenuity to devise his own projects should probably not try to use the method.

The whole thing is a reminder of the days of our youth when in the malarial time of the year certain sugar-coated, pink pills were administered to improve our health. The pink coating tasted fine, but the pill—alas! The pill was necessary to the health though it caused a wry face if not swallowed before the pink coating dissolved. The taste of the pink coating "motivated" the act of pill swallowing. Many of the schemes for motivation in the learning process are but sugar coating on the educational pill. It is legitimate deceit. The pupil does not like the pill sometimes but should take it; hence the sugar-coated motivation which, by the way, reaches back to and beyond the time of Petalozzi. But, should all education be administered surreptitiously? Shall we not also cultivate gradually in the pupil (especially in the high school) an ambition to do things for the sake of improving himself which he may not enjoy above his play, leisure, or social life. Let him see the real goal once in a while! Projects, or any method of motivation, should be used whenever they lend themselves harmoniously to the orderly development of the course but there will be intervals when the only motivation is a feeling that "other folks are learning these things, so must I". When that interval is so long (perhaps a bit of mechanical algebra) that the pupil must make a conscious effort to learn something which makes little or no appeal to him there is a loss of efficiency in the learning process, to be sure. An ingenious teacher will make those intervals as short as possible.

The track engineer of an electric railroad that gets its power from a third rail is forced to leave gaps in the third rail at switches, crossovers, etc. He makes these gaps as short as possible and the momentum of the train carries it over the gaps until contact is again made with the motive power. So may the ambition to learn, inspired possibly by experience in projects of practical usefulness of the subject, carry the pupil over the stretches of mathematics where the subject matter itself makes no inspiring appeal to him to urge him on. Such a use of projects seems more reasonable than to subordinate the development of a course like algebra or geometry to a method of presentation.

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CHAPTER VIII

IMPROVED METHODS AND MODERN TENDENCIES

68. There Is No One "Best Method". Some teacher may have become so enthusiastic over the possibilities of the project method that he is sure it is a cure for all the shortcomings of our mathematical teaching. He thinks all teaching should be molded to fit this method. Another teacher may claim the same virtues for some other method, and so on. The very fact that different ones are so sure about different methods, each being successful with his own method, is sufficient proof of the fact that there is no method to be used to the exclusion of others.

In the columns of the newspapers we find advertisements of the "patent" medicines, many a one of which is guaranteed to cure an amazing variety of diseases. It simply will not do it. The ingredients of the medicine may fit the case of some individual and he writes an honest "testimonial". He wants all his neighbors and friends to try it. If their cases are different they may be very much disappointed. So it is with a method of teaching. It may fit one teacher's personality and be a failure in the hands of another. This consideration leads to the question discussed in the next section.

69. Why Several Methods? There are three elements entering into the teaching of a class; the teacher, the class, and the subject taught. The method of teaching is a function of these three things. If they are constants (same teacher, same class, same subject) the method may remain unchanged. But, as a function of one or more variables is a variable itself so will the method of teaching vary if we change either the teacher, the class, or the subject.

Two teachers, unless they are very much alike, can rarely use the same methods of instruction. One teacher may have the knack of asking thought-provoking questions so that the heuristic method with him is a powerful tool; another may be a complete failure as a questioner and because of it may succeed better with any other than the heuristic method.

A good teacher will seldom use exactly the same methods in teaching the same subjects to different classes. If he finds that he can, it is because the classes are so much alike in nature that the same conditions present themselves in both cases. This is sometimes true in a large school where several sections in the same subject are enrolled without any attempt to classify the pupils. Any good teacher who has taught for several years knows what it means to try in vain to get from certain classes the responses to his methods to which he has become accustomed in other classes. A variation in method becomes absolutely necessary.

The teacher who finds himself following the same outline for a course year after year and presenting the material in the same way at every stage of progress, irrespective of the kind of pupils in the class, had better analyze his "patent method". Let us take as a notable example, the teaching of geometry. It has been a tremendous battle to free the teaching of geometry to fifteen year old high school pupils from the traditional methods used when it was taught to more mature scholars. While the run of classes in the same school may not differ so much in maturity, there are minor differences that any good teacher should observe, differences of sex, major interests, home or community environments. We must not have such iron-clad methods that we can not change to fit the varying needs.

It should not be necessary to mention that a teacher can not most successfully use exactly the same methods in teaching geometry as in teaching algebra. For example, the element of drill necessary to a certain degree to develop at least a minimum required proficiency in mechanical algebra is not so necessary in geometry. The recitation method would probably be used more by the average teacher in algebra than in geometry, the laboratory method probably more in geometry than in algebra, though each method belongs in both.

Finally, a teacher as a rule can not copy the methods of another, however good they may seem and however successful the other teacher may be in the use of them, without adaptation. The personality of one teacher will make a certain method successful while the personality of another teacher is best fitted for some other method. To see some teacher in his attempt to use without adaptation or variation, a particular method of some successful teacher reminds one of a small person wearing a large person's coat. It does not fit.

Sometimes young teachers, just out of college, unconsciously adopt the methods of their college professors, whom they acknowledge as master teachers, not realizing that those same college professors would not use the same methods in teaching high school mathematics to boys and girls.

The method should be adapted to fit the teacher, the class, and the subject matter.

70. A Working Combination. What shall an inexperienced teacher do when he has had no chance to develop methods of teaching? He must use the methods of others, but he should choose standard methods rather than the particular methods of individuals. He should not confine himself exclusively to any one method. Of the four methods mentioned in Chapter VI as pertaining to the subject matter there is no doubt that in geometry the inductive and analytic should be stressed because the usual methods of a text over-emphasize the deductive and synthetic. So much of algebra is deductive that the inductive method should also be emphasized there in the development of rules and formulas. For example, the law of exponents governing the product of powers of a base may first be seen by using particular numbers, say, $x^2 \cdot x^3 = (x \cdot x)(x \cdot x \cdot x) = x^5$, etc., finally reaching the generalization, $x^m \cdot x^n = x^{m+n}$, from which all future exercises follow by deductions.

Concerning methods of presentation, every teacher should from the beginning of his career guard against the "lecturing" habit. He should overcome the lazy habit of the recitation method. It is true that the recitation method must be used some since our schools demand that we grade the pupils according to what we think they have learned. There should be as little of the pure recitation method (where the teacher gives nothing, but simply takes information from the pupil) as possible. Most of the necessary information concerning a pupil's growth in knowledge can be obtained while using the heuristic method so that the recitation method will be needed very little. Project teaching is a possibility for any teacher who can instill the true spirit of investigation that should attend it. The projects should be fundamentally connected with the subject matter and carry on the purposes and aims of the course. They should certainly not dominate the course. The laboratory method (not necessarily confined to the handling of material equipment) which is so closely allied to the project method should be used every time possible. Both these methods work well in harmony with the heuristic method which, more than any other method, every teacher should use as much as possible if he can do it successfully. The heuristic attitude can not be acquired in a day. If a teacher does not have it he should study the method and try to catch the spirit of it. He should discard the attitude of a knowledge disseminator and cultivate the attitude toward his pupils of a helpful leader who guides them into the proper lines of thought where they seek and find the desired information.

71. **Supervised Study.** One of the new features in the organization for the instruction in the schools of the twentieth century is the plan of supervised study. The usual length of the recitation is doubled, about half the time supposed to be devoted to recitation and half to supervised study. This does not mean that the teacher conducts a recitation for half the double period and then sits at her desk and keeps order, answering occasional questions, during the remaining half while the students presumably prepare the lesson for to-morrow but in all probability (under such management) would be studying for the next immediate recitation. It does not necessarily mean that the teacher of mathematics uses the second half of the period to move about the room answering signals of distress and explaining to individuals how to solve hard problems or by inspections over shoulders determine whether this pupil or that is on the right track. It does not necessarily mean that the period must be set off into first and second parts, kept invariably distinct. It should mean that the whole period assigned to mathematics is to be given to mathematics and that not more than half of it, approximately, is to be used for the same purpose for which the shorter recitation period of the past was used. Just how that half of the period is to be distributed depends on the class, the teacher, and the nature of the subject matter at the time. It gives the teacher a wonderful opportunity to use the laboratory method, the project method, and the inductive method. Time would allow the development inductively of such a theorem as that about the sum of the interior angles of a convex polygon used for illustration in Chapter VI; or for paper folding and cutting methods as conducive to the discovery of proofs. For lessons on original exercises in geometry, or verbal problems in algebra, the pupils could be guided into the right lines of thought and left to set down the work and draw the conclusions as a part of their individual study for the next day's recitation. No longer should a pupil be placed in the predicament of facing a set of problems without a clue and, after an hour of unsuccessful and discouraging attempts, giving up in despair.

As an example of the use of a supervised study period in algebra, let us suppose that the class is ready to take up the study of the solution of quadratic equations by completing the square. Under the old plan of recitation the teacher might have five minutes (maybe ten, but doubtful) to explain the new subject before dismissing the pupils to learn it if they can and solve a certain number of equations. The customary result is a half understood, memorized rule on the part of the pupil with various degrees of success or failure. The new concepts included in the teachers' explanation must meet the interference of the other recitations and other teachers' explanations of other things before the pupil has a chance again to concentrate his mind on quadratic equations. Under the plan of supervised study the teacher might assign a series of exercises like the following[54]:

$x^2 = 4$
$x^2 = 7$
$(x + 1)^2 = 9$
$(x + 3)^2 = 10$
$x^2 + 2x + 1 = 16$
$x^2 + 8x + 16 = 2$
$x^2 + 2x = 8$
$x^2 + 6x = 1$
$x^2 + 2x + 15 = 0$
$x^2 + 8x + 14 = 0$
$(x + a)^2 = b^2$
$x^2 + 6ax = 3a^2$

By wisely guiding the work of the class for twenty minutes or so the teacher can fasten in the pupils' minds the facts that:

(1) If the unknown is squared in the equation, a square root must be taken to find its value.

(2) If the unknown enters in the quadratic form a square root must be taken to reduce the equation to a linear form.

(3) If a perfect square is not obvious one must be formed.

(4) The perfect square which contains both x^2 and x is the trinomial square.

The purpose of every step of the process is opened up before the pupil and he attacks the next day's problems with a feeling of mastery and understanding rather than with a nervous haste to find what the book did in the illustrative example and do just like that. He has had time to work with the new principles under the teacher's guidance and so fix them in his mind that they may not be easily erased.

This also illustrates the inductive method of teaching (see §57). In the next lesson the pupil can be led on to the solution of $ax^2 + bx + c = 0$ and thus have the general formula.

Other topics in algebra and in geometry would demand other ways of conducting the supervised study. There certainly should be no iron-clad rule governing it.

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[54] See Young, *The Teaching of Mathematics*, p. 108, for a similar set.

72. **The Report of the National Committee on Mathematical Requirements.** In 1916 the Mathematical Association of America organized a committee for the purpose of studying the problem of improvement of mathematical education in the secondary schools and colleges. The committee was first composed of six nationally known college professors of mathematics. Later, at the request of the original committee, representatives of the secondary schools were added, the membership ultimately consisting of six university and seven high school representatives. The first findings of the committee were published as tentative reports and discussed at various meetings of organized groups of mathematics teachers throughout the country, criticism being invited. In this way a very great many of the progressive teachers of secondary mathematics in the United States had a part in the work of the Committee. The Committee states in the preface to its final report that it believes "that the recommendations of this final report have the approval of the great majority of progressive teachers throughout the country". The final report, covering more than 600 pages, entitled, "The Reorganization of Mathematics in Secondary Education", was published in 1923 and until recently a copy could be had by any teacher of mathematics by sending to the Dartmouth Press, Hanover, N. H., enclosing 20 cents to pay for postage and wrapping. The supply is now (March, 1926) exhausted. If funds are secured for another edition it may again be possible to secure a copy. The contents of the "Report" are given here:

Part I. General Principles and Recommendations

Chapter I. A brief outline of the report.

Chapter II. Aims of mathematical instruction—general principles.

Chapter III. Mathematics for years seven, eight, and nine.

Chapter IV. Mathematics for years ten, eleven, and twelve.

Chapter V. College entrance requirements.

Chapter VI. List of propositions in plane and solid geometry.

Chapter VII. The function concept in secondary school mathematics.

Chapter VIII. Terms and symbols in elementary mathema

Part II. Investigations Conducted for the Committee.

Chapter IX.—The Present Status of Disciplinary Values in Education. By Vevia Blair.

A critical survey of the scientific literature relating to disciplinary values and the transfer of training, followed by an attempt to formulate conclusions warranted by the results of investigations and supported by leading educational psychologists.

Chapter X.—A Critical Study of the Correlation Method Applied to Grades. By A. R. Crathorne.

An investigation to determine the serviceability and reliability in educational problems of the theory of correlation applied to grades. The chapter begins with a nontechnical description of the theory of correlation. This is followed by an attempt to formulate a scale of values for the correlation with a view of determining with as much precision as possible what constitutes "high," "medium," or "low" correlation. Finally the theory is applied to all the studies in the high-school curriculum—with some significant results as to the educational influence of certain groups of subjects as compared with other groups.

Chapter XI.—Mathematical Curricula in Foreign Countries. By J. C. Brown.

A brief account of the work in mathematics offered by the elementary and secondary schools of Austria, Belgium, Denmark, France, Germany, Holland, Hungary, Italy, Japan, Roumania, Russia, Sweden, and Switzerland, followed by a general summary comparing conditions in the United States with those in the countries mentioned.

Chapter XII.—Experimental Courses in Mathematics. By Raleigh Schorling.

An extensive description of the work in mathematics given in 14 leading experimental schools of this country—giving in detail the order of topics and the time devoted to each, the equipment of the schools, the cost of the instruction, etc.

Chapter XIII.—Standardized Tests in Mathematics for Secondary Schools. By C. B. Upton.

A description of various standard tests in arithmetic, algebra, and geometry, and a discussion of their use in the teaching of mathematics. The tests considered are the following: The Courtis arithmetic tests, the Woody arithmetic scales, the Woody-McCall arithmetic test, the Stone reasoning tests, the Courtis standard practice tests in arithmetic, the Studebaker practice exercises in arithmetic, the Rugg-Clark practice exercises in first-year algebra, the Hotz first-year algebra scales the Kelly mathematical values test, the Minnick geometry tests, the Rogers prognostic tests of mathematical ability, and others.

Chapter XIV.—The Training of Teachers of Mathematics. By R. C. Archibald.

A selection from the results of an extended and detailed investigation, covering every State in the Union and all the larger cities, concerning present conditions regarding the training of teachers of mathematics, the facilities for providing such training, and the requirements for certification. The selection is made for the purpose of exhibiting the highest standards to be found in various parts of the country and the courses of study for the training of teachers given in various institutions. The chapter also includes a brief survey of conditions in certain foreign countres.

Chapter XV.—Certain Questionnaire Investigations. By W. F. Downey, A. R. Crathorne, Alfred Davis, and others.

These investigations relate to the interests of high-school pupils, change of mind as to life work between high school and college, the importance of mathematics, etc.

Chapter XVI.—Bibliography on the Teaching of Mathematics. By D. E. Smith and J. A. Foberg.

This bibliography lists all the articles relating to mathematics that have appeared since 1910 in a number of leading educational periodicals, and gives in addition to author, title, and place of publication a brief summary of each article.

73. **Modern Views on Transfer of Training.** No subject relating to education has been so thoroughly discussed, with such bitterness at times and with such widely opposing views, as the question of transfer of training. Will certain skills formed in one subject carry over into another? Will ability to reason well in the proofs of geometric theorems insure successful reasoning by the person in the problems of later life? The old theory of formal dicipline insisted that a pupil should study "solid" subjects that demanded much brain exercise if he would best prepare for success in later life. His study was for the sake of the mental discipline rather than for the utilitarian value of the subject matter. A reaction against this theory set in and a decade or so ago many leading psychologists denied the existence of any transfer of training. Some of them would have practically nothing taught that did not contribute directly to the student's "bread and butter" ability. The National Committee conducted an investigation, collecting the opinions of 24 leading psychologists by means of a questionnaire (Report, Chap. IX). From the answers of the psychologists the committee concludes (Report, p. 95) that "the two extreme views for and against disciplinary values practically no longer exist"; that "transfer does exist"; that "negative transfer exists" but may be avoided by proper methods of training; that very few experiments have shown the full amount of transfer because of the imperfections of the experimental setting; that "the amount of transfer is very largely dependent upon methods of teaching"; that "transfer of training (with certain restrictions) is a valid aim in teaching"; and that "transfer is most evident with respect to general elements—ideas, attitudes and ideals".

John R. Clark of Teachers College, Columbia University, says[55]

> "Ninety percent of the recognized authorities in psychology today would probably approve the following propositions:
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> 1. In itself the transfer value of mathematics (or of any other subject) is insufficient to justify its being required of all pupils.
> 2. The amount of transfer from mathematics depends upon the method of teaching the subject.
> 3. Meanings, methods of attack, and attitudes are more transferable than skills and information".

The modern movement toward reorganization of mathematics does not assume no existence of transfer, and therefore does not attempt to eliminate all from the course except the easy and utilitarian parts. Those concerned with the reorganization do maintain that:

(1) **The methods of the part are not so perfect that they may never be changed;** for to believe they are means stagnation, not progress. Surely each quarter of a century—even each decade—produces a corps of teachers who are capable of improving on the methods of those gone before.

(2) **Traditional belief in the mental discipline theory has had too much influence** tending to keep invariant the content of the courses in secondary mathematics and the methods of teaching.

It is the opinion of the writer that we can not proceed too cautiously with reforms. Many educational theories are being tried out (as they have always been) that are heralded by their sponsors as solutions of this and that. Some of them will be forgotten in ten years. It is the function of educational experts to discover and experiment with new theories but it is the duty of a teacher to choose established methods, unless the teacher be capable of educational research. For the average teacher the quotation,

"Be not the first by whom the new is tried,
Nor yet the last to cast the old aside."

applied to methods is very apt.

The two following statements are taken from page 11 of the Report of the National Committee:

> *The primary purposes of the teacher of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment*

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[55] *Mathematics in the Junior High School,* Gazette Press, Yonkers.

and to appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.

These are the statements of a group of teachers of mathematics ranging from experienced high school teachers to university professors known nationally as research mathematicians and, to say the least, should be seriously considered. "Habits of thought and of action" are strengthened by repetition of the acts. Nowhere in all the range of secondary education is there a subject which demands so much repetition of the act of warily looking out for all conditions, summing them up, and striking out independently for a conclusion as in the course in plane geometry, if properly taught. The habit of independent thinking rather than blind following of the dogmas of authority is a very valuable one to cultivate in our future American citizens. It is needed sorely at present in politics, in business, in education, and in religion. Whether this developed habit becomes "effective in the life of the individual" depends upon whether the teacher presents geometry in its relation to ordinary life or as an abstract branch of mathematical science only for the sake of so-called mental discipline. Modern educational theory demands the former method rather than the latter. Finally; habits "carry over", for they are reasonably permanent in the life of the individual.

74. **The Junior High School Movement.** Every educator is now thoroughly familiar with the psychological foundations for the recent widespread movement to reorganize our system of education, making the elementary school cover the first six grades and the secondary school the next six. The ideal plan is to subdivide the six years of the secondary school into the junior high school (covering grades seven, eight, and nine) and the senior high school, (covering grades ten, eleven, and twelve.) In many places where the present building equipment does not accommodate itself to a 6—3—3 plan, the organization is made on a 6—2—4 plan. In the latter case the seventh and eighth grades in the usual 8-grade elementary school are organized for instruction by the teacher-subject method instead of the teacher-grade method, the recitation periods are lengthened and various

other changes made in order to adapt the instruction to the pupil's development and to present a worthy challenge to his rapidly growing powers of thought.

The effect of this organization on the mathematics courses of grades seven, eight, and nine is very marked. Since the end of the ninth grade (graduation from junior high school) would mark a possible stopping place in the educational career of so many pupils, the new courses are planned so as not to be merely preparatory courses for future mathematics.

The National Committee publishes (Report, p. 14) this statement:

> In the junior high school, comprising grades seven, eight, and nine, the course should be planned as a unit *with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little referenec to courses which he may or may not take in succeeding years.*

There is no valid reason why this purpose stated above can not be accomplished without diminishing in any way the preparation of the pupil for future mathematics. A detailed discussion of the courses is presented in the next chapter.

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CHAPTER IX

THE REORGANIZED MATHEMATICS CURRICULUM OF THE JUNIOR HIGH SCHOOL

75. The New Subject Matter. On page 22 of the Report of the National Committee we read:

> The amount of time devoted to arithmetic as a distinct subject should be greatly reduced from what is at present customary. This does not mean a lessening of emphasis on drill in arithmetic processes for the purpose of securing accuracy and speed.

The following personal reminiscence will illustrate this:

As a boy I studied house-papering and plastering in arithmetic. Various rules (hard to apply) were learned and several imaginary houses papered and plastered. Later I had occasion to contract for the services of paper hangers and plasterers and discovered to my surprise that neither class of workmen knew or used the rules I had learned in arithmetic. With extreme accuracy I had computed the deductions due for opening in my house before the plasterer came only to have him tell me that for ordinary windows and doors no deduction was made. This represents one type of material the omission of which is suggested. The method of extracting cube root represents another type.

To replace the omitted parts of arithmetic in the seventh and eighth grades are suggested graphs, equations, intuitive geometry, approximate computation and the use of tables, and the history of the development of mathematical processes. We shall discuss these subjects in turn.

76. Graphs. No better subject than graphs could be chosen to illustrate the manner in which modern life demands a change in the teaching content of arithmetic. Twenty-five years ago the average person needed no knowledge of graphs in his ordinary reading. If statistics showing the trend of prices were presented, or a comparison of the standing armies of the different nations made, or a report on the distribution of the tax money to different departments of the government printed, he saw it all in columns of figures. Now he would see the trend of prices represented by a broken line graph showing rise and fall. He would probably see the standing armies compared by means of a pictogram representing by the heights of soldiers (each in the uniform of his nation) the relative sizes of the standing armies;

or, which is not so misleading, by means of a bar graph, that is, a series of shaded lines representing by their relative lengths the sizes of the various armies. He would probably see the relative costs of the various departments of government represented by a circle graph where sectors of a circle show relative amounts disbursed by the departments, the whole circle representing the entire tax collected and disbursed. The circle might represent a dollar and the graph be entitled "Where the taxpayer's dollar goes". Not all of these graphs which strike the eye of the reader are easily interpreted by one of no experience with them. They must be understood if he is to read intelligently. The conclusion is obvious—pupils must be taught in school the meaning of these things. In no other course does this subject matter fit more aptly than in the course in mathematics.

In recent years it has been repeatedly pointed out that relations between quantities play a great part in our lives. The National Committe says (Report, page 12):

> The one great idea which is best adapted to unify the course is that of the functional relation. The concept of a variable and of the dependence of one variable upon another is of fundamental importance to every one.

The standard method of picturing a functional relation (say, the relation expressed by some practical formula) is by means of the continuous graph. Chapter VII of the Report of the National Committee is devoted to the discussion of the functional relation and Chapter X of this bulletin takes up in detail the same subject. In Chapter XI of this bulletin will be found a detailed discussion of the teaching of graphs in algebra.

77. Equations in the Seventh and Eighth Grades. The traditional plan of teaching arithmetic, algebra, and geometry in distinctly separate courses has for too long a time robbed the student in arithmetic of his right to use that powerful tool of mathematics, the equation, to simplify the solution of his problems. Of course there were many teachers of the seventh grade in the past who did not know how to teach equations—who did not even know anything about equations. (Let us hope there are none now.) This has probably had much to do with holding back all instruction in algebraic topics until the ninth grade. The present junior high school organization and policy of administration demands higher standards for teachers. We should now expect that the average junior high school teacher has studied some college mathematics. This condition removes one of the strongest reasons why equations have not been taught in the earlier grades.

The use of an equation to solve a problem like the following (taken from a recent text for the seventh grade) will be welcomed by the seventh grade boy or girl:

> Three brothers, James, Henry, and Sam, agreed to pick a farmer's peaches for \$7.50, the money to be divided according to how many peaches each boy picked. James picked twice as many bushels as Henry, and Sam three times as many as Henry. How much did each get?
>
> If we make use of letters for unknown numbers and use the equation, we have the following solution.
>
> *Solution.* Let $n =$ the number of dollars Henry received.
>
> Then $2n =$ the number of dollars James received, for he picked twice as many bushels as Henry.
>
> $3n =$ the number of dollars Sam received.
>
> Since the sum of the number of dollars they received was 7.50,
>
> $$n + 2n + 3n = 7.50$$
>
> If now we replace the left member by the sum of the n's, we have
>
> $$6n = 7.50$$

The remaining part of the solution is obvious and is omitted here to save space.

It goes without saying that the use of literal symbols and the primary ideas of solving equations would be introduced by means of simpler problems than this one. This problem occurs near the end of a list and is preceded by easier ones.

The use of the equation in percentage eliminates many of the difficulties of the course of the past. This will immediately raise the question, "Is the purpose of the new content in the course to make the course easier? Are we to rob the pupil of his training in arithmetical analysis?" It certainly is not meant to make the course as a whole easier just for the sake of relieving the pupil of hard study but it is meant to make parts of it easier in order that the pupil may thereby have time to spend on valuable topics which have heretofore been unknown. Nothing in mathematics should be made difficult just for the sake of being difficult. There is enough difficult material to satisfy the demands of the most hardened "disciplinist" even after we simplify many parts by means of the equation. On the other hand, nothing that should properly be taught at the time should be omitted just because it is difficult. By using the equation to abbreviate and simplify the solutions of problems more ground can be covered, and the pupil's knowledge of practical facts and principles can be extended. There are many other opportunities for exercising the pupil's reasoning powers which can replace the problem analysis eliminated by the equation method. To decline to use the simplest method of solution of a problem just to make the problem an exercise in mental discipline is not fair to the pupil and is not true to the spirit of mathematics.

The equation can be advantageously used in percentage, the pupil being led up to it through ratio. The following problems are types:

What number has the ratio to 25 that 16 has to 100?

Solution. Let n = the number, then

$$\frac{n}{25} = \frac{16}{100}.$$

* * * * * * * * *

Out of a class of 50 pupils 40 were present. What was the number, or rate, per hundred present?

Solution. This might be as follows. The ratio of 40 to 50 is equal to the ratio of what number to 100?

This statement would be expressed by the equation

$$\frac{40}{50} = \frac{n}{100},$$

where n = the number, or rate, per hundred.

* * * * * * * * *

It is important to notice that per cent always refers to per cent (or number of hundredths) of *some definite amount.* Frequently the definite amount is not stated. For example, per cents obtained by students in tests, such as 95%, 92%, 85%, etc., do not necessarily tell us how many examples or questions were given. Such per cents, however, do give us a means of comparing the results.

EXERCISES

1. What per cent of 25 is 3? 12? 17? 23? 25? 28? 50?

Solution. These exercises can be done orally if you can think through the following steps. In the more difficult ones use the equation method.

ORAL METHOD	EQUATION METHOD
$\frac{3}{25} = \frac{?}{100} = ?\%$.	$\frac{3}{25} = \frac{n}{100}$.
$\frac{3}{25} = \frac{12}{100} = 12\%$.	Multiply each member by 100: $12 = n$
Check. $\frac{3}{25} = \frac{12}{100}$.	Ans. 3 is 12% of 25.

In addition to the types of problems already mentioned the equation can be used in solving interest problems and in dealing with the formulas of mensuration. All of the material mentioned in this section is taught in the seventh grade.

78. **Intuitive Geometry.** This subject has always been taught to some extent in connection with mensuration. The method of presentation in such connection, however, usually deprived it of any real value except for the information imparted which was necessary to understand mensuration.

There are reasons for teaching intuitive geometry other than for the purpose of learning mensuration. The subject of graphs demands an understanding of elementary geometric forms. Drawing to scale, or mapping, demands fundamental ideas of similar figures. A pupil may get a fair understanding of the meaning of house plans by drawing to scale the floor plans for houses. A problem in mensuration grows out of the necessity of excavating for the basement. If bay windows are in the plans, an application of the formula for the area of a trapezoid is found.

Since the use for this knowledge of geometry is found in more than one chapter, it should be taught not as an introduction to any one subject, like mensuration, but as a separate topic. Instead of giving the pupil in "tabloid" form certain "doses" of definitions of geometric forms he is allowed to acquire this information by experience with these forms, for example, by drawing right angles, parallel lines, perpendiculars, circles, squares, etc., with the ruler and compasses. By experimental exercises he can be led inductively to the assumption of many of the theorems used in mensuration. If he be properly taught that he only assumes on the basis of his experiments the truth of the geometric statements that can be demonstrated later, no harm is done and the fact that he can see a foundation for believing they are true makes him appreciate more the real meaning of the mensuration formula than to have it set down merely to be memorized and used.

For illustration, let us take the formula for the area of a triangle. The pupil can be led to see that the triangle is half a parallelogram having the same base and altitude by cutting the parallelogram apart and comparing the parts. Or, he can see that the triangle is half a rectangle having the same base and altitude. Though he can not prove the equality of the triangles used in the development, he can make the other steps in the reasoning as safely as he might in demonstrative geometry. Granting that the area of a rectangle is the product of the base by the altitude, the area of the triangle easily follows as half that product. Again, if he knew the formula for the area of a triangle, the formula for the area of a trapezoid may be developed deductively (if desired) just as in demonstrative geometry.

Do not understand this illustration to mean that the deductive proof is the goal of intuitive geometry. Very little of the work would lead to deductive exercises. It merely shows how it is possible to lead the pupil "behind the scenes" and show him what is back of some of those (sometimes) arbitrarily given formulas. It shows how he may be given a broader and more fundamental conception of the formulas. The deductive type of work should not be done before the ninth grade, though intuitive geometry may be first introduced by measurement exercises in the seventh.

79. **Approximate Computation and the Use of Tables.** The National Committee has the following to say on this subject (Report, page 28):

> The solution of problems should offer opportunity throughout the grades under consideration (seventh, eighth, and ninth) for considerable arithmetical and computational work. In this connection attention should be called to the importance of exercising common sense and judgment in the use of approximate data, keeping in mind the fact that all data secured from measurement are approximate. A pupil should be led to see the absurdity of giving the area of a circle to a thousandth of a square inch when the radius has been measured only to the nearest inch. He should understand the conception of "the number of significant figures" and should not retain more figures in his result than are warranted by the accuracy of his data. The ideals of accuracy and self-reliance and the necessity of checking all numerical results should be emphasized. An insight into the nature of tables, including some elementary notions as to interpolation, is highly desirable. The use of tables of various kinds (such as squares and square roots, interest, and trigonometric functions) to facilitate computation and to develop the idea of dependence should be encouraged.

What sense is there in giving the dimensions of a cylindrical cistern as 8 feet in diameter and 16 feet deep and expecting the number of gallons it will hold to be computed as follows (volume in cubic inches $\div$ 231):

$3.1416 \times 48 \times 48 \times 192 \div 231 = 6016.2048$; Vol $= 6016.2048$ gal.

Such methods have been approved in text books of the past and such an answer has been recorded in the book.

In the first place, who would want to know the answer to one ten-thousandth of a gallon? In the second place, and more vital, was there ever a cylinder made as big as a cistern with an accuracy such that an answer like that above could be depended upon? In actual practice (and for what else do we teach practical mensuration) the dimensions of a cistern may vary by inches or more. The average diameter of the cistern might not be expected to be exactly 8 feet. A more reasonable set of dimensions would be 8 ft. 2 in. in diameter and 15 ft. 9 in. deep, (meaning the approximate average values) or better, 8.2 ft. in diameter and 15.8 ft. deep. Thus the data given in the problem should suggest "the fact that all data secured from measurement are approximate". In that case the value 3.14 for π would be more in keeping with the accuracy of the supposed measurements. Finally, considering still the answer to the above problem, even if a perfect cylinder eight feet in diameter and sixteen feet deep is contemplated, can the last three figures in the answer be credited on the basis of the value of π used? This fault will be made clear in the following quotation from a text in junior high school mathematics (eighth grade) which teaches a reasonable method of dealing with data secured from measurement.

Approximate Products. When numbers that are derived by measuring are met in computation, the degree of accuracy obtainable in results depends upon the kind of quantity measured and upon the precision with which the measurements are made—not upon the number of figures actually used in performing the computation. A result can be no more accurate than the least accurate measurement entering into the computation.

Note. Commercial problems dealing with dollars and cents should not be confused with problems dealing with numbers obtained by measurement. In a commercial problem, when the number of units is known and the price per unit is given, the result should be computed to the nearest cent.

Example 1. The length of a room is 37.4′, and its width is 23.8′. Find the area of the floor in square feet.

Formula. $A = bh.$

$A = 37.4 \times 23.8$

When we measure the length of a room and record it as 37.4′, we generally mean that the true length is nearer to 37.4′ than it is to 37.3′ or to 37.5′. The length might be nearly 37.45′ or slightly more than 37.35′ Similarly, we might record the width as 23.8′, meaning that it represented the width to the nearest tenth of a foot. We shall find that the area computed from these measurements will not be dependable beyond the nearest square foot.

Specimen Solutions.

A. The work appears as follows if the customary method of multiplication is used:

$$\begin{array}{r} 37.4 \\ 23.8 \\ \hline 2992 \\ 1122 \\ 748 \\ \hline 890.12 \end{array}$$

Estimate:
$20 \times 40 = 800$

The area of the floor appears to be 890 square feet to the nearest square foot.

B. If the order of the multiplying is reversed; that is, if 37.4 is multiplied by the left-hand digit of the multiplier first, the work will appear as follows:

```
  37.4
  23.8
 ------                                   Estimate:
 748.        (20 × 37.4)              20 × 40 = 800
 112.2       (3  × 37.4)
  29.92      (.8 × 37.4)
 ------
 890.12
```

Explanation. To obtain the first partial product, the figure 2 of the multiplier is used first. Now 2 here represents 2 *tens*, that is, 20; so we place the decimal point in this first partial product after the 748 (estimate was 800).

To obtain the next partial product we multiply by 3, placing the figure first obtained in this partial product one column to the *right* of the preceding product.

To obtain the last partial product we multiply by 8, placing the figure first obtained in this partial product again one column to the *right* of the preceding product.

In this method we locate the decimal point by estimate and common sense early in the work, that is, in the first partial product. Carelessness and errors in its location in the final product are thereby prevented.

C. The measurements of the floor of which we are finding the area were made only to the nearest tenth of a foot. It will now be shown that the area can not be accurate beyond the third figure and that even that figure is in doubt.

```
  37.4
  23.8
 ------                                   Estimate:
 748.**                               20 × 40 = 800
 112.2*
  29.92
 ------
 890.12
```

The area of the floor is computed as 890.12 square feet. In the first partial product, however, the places marked with stars (*) are not necessarily zeros. Had the length been measured to the nearest hundreth of a foot, some figure would have appeared after 748 in the column following the decimal point. The stars show that the figures in these columns are doubtful. In fact, the first partial product, 748, might be either 747 or 749, for 20 times 37.35 is 747 and 20 times 37.44 is 749 to the nearest third figure. (Any measurement between 37.35′ and 37.45′ would be called 37.4′, to the nearest tenth.)

In the final product, therefore, the figures that appear in the doubtful columns (which happen, in this example, to be at the right of the decimal point) are of little use, as their retention does not make the answer any more accurate than it would be without them. This final product might be as small as 887.1 (37.35 × 23.75) or as large as 892.5 (37.44 × 23.84), depending upon the data. Hence the *third* figure of this answer is in doubt and we waste labor by keeping figures in the partial products that do not appear in our final product.

D. If we cut the partial products, by not writing any figures which would come in the doubtful columns, we shall obtain the figures necessary for the final product. The work is shown below:

$37'.4'$
$23\ .8$
748 .
112 . ($3 \times 37 + 1$. See Note 1.)
30 . ($.8 \times 30 + 6$. See Note 2.)
890 .

Estimate:
$20 \times 40 = 800$

Ans. $A = 890$ sq. ft.

Note 1. ($3 \times 37.4' = 112.2$). As the .2, if written, would come in a doubtful column (see Solution *C*), we do not write it; but in multiplying 37 by 3, we note that we have 1 to carry (since $3 \times .4 = 1.2$, which is nearer 1 than 2). We now place a mark (′) over the 4 in the multiplicand. This is to show that we have not written a figure in the partial product that would come in a doubtful column, but that we have noted, in multiplying, its effect upon the next column to the left.

Note 2. ($.8 \times 37'.4' = 29.92$). As the 9 and the 2, if written, would both come in doubtful columns (see Solution *C*), we do not write them; but in multiplying 30 by .8, we note that we had 6 to carry (since $.8 \times 7 = 5.6$, which is nearer 6 than 5). We now place a mark (′) over the 7 in the multiplicand to indicate again that we have not written a figure that would come in a doubtful column; but that we have noted, in multiplying, its effect on the next column to the left.

Note 3. Measurements like the above, made to the nearest third figure, give a result not reliable beyond three figures; hence figures beyond the third may be neglected. Even the third figure may not be reliable. If the first three figures of a result are reliable, such a result is said to be of *three-figure accuracy.*

An approximate method for division is developed in a similar manner. The uselessness of continuing division beyond the possibility of accuracy can be appreciated by the reader if he will try finding the diameter of, say a cylindrical column whose circumference (as measured with a tape line) is 123.5 in. Use $\pi = 3.14$. After the first three digits in the quotient are secured, the succeeding remainder-dividends are incorrect (since subtracting 3.140000 in the first step is not the same as subtracting 3.1416 — or 3.141593) and consequently all figures in the quotient after the third may be wrong.

It is of more value to a student to be trained to think along lines of common sense in problems of computation from data secured from measurement, having a method of approximate computation which saves him time, than to be "figure perfect" in long, tedious, and sometimes useless, computations.

80. History of the Development of Mathematical Processes. In the earlier chapters of this bulletin some discussion was made as to the value to the high school student of the knowledge of the history of the mathematics he studied. Arguments made there hold also for the student of the seventh or eighth grade. Why should we not at the beginning of the seventh grade, when the general principles of the Hindu-Arabic notation are summarized, give the pupils an insight into the origin of these numbers in some manner similar to the following introductory pages of a recent seventh grade text?

> Do you know that ages ago people did not know how to count? No one could count up to ten, for nobody had made any names for numbers. The people were not civilized as we are—they did not even have a system of reading and writing. Finally some one thought of making a symbol for a man which resembled a man, a symbol for sunrise which was a semicircle on a horizontal line, and symbols for other things they knew about. From this grew up a kind of picture writing. The American Indians used such picture symbols.
>
> If a man sent a message concerning two men or three men, he drew two symbols like men or three symbols like men to represent the men. Suppose he wanted to say that some event had happened three times. He soon learned to write three straight marks (|||) to stand for three as it does to-day on the faces of some clocks, as chapter numbers in books, and in various other places.
>
> Imagine a boy who lived long ago who had caught five fishes. He tries to count them. He touches them one by one, saying some words like our one, two, three, and then he stops; but there are more to count. He cannot count them. He may say, "One, two, three, *many*." If some one asks him how many fishes he has, he may say, "Many," or he may hold up five fingers. Did you ever answer a similar question by means of your fingers?
>
> Number names were finally devised up to five, and the notion of hand was connected with five. In the Russian language the words for "fist" and "five" differ by one letter and sound very much alike. The Kiowas and kindred tribes in Oklahoma used a picture of a hand to stand for five. People who counted by "hands" (fives) learned to say "hand and two" for seven and "hand and three" for eight, etc., just as we have "ten and three" represented in the symbol 13. But they could only count up to "five hands and five," or 30. Any greater number was "many." Next came the idea of using both hands. The reason our number system is based on ten is probably because we have ten fingers. Suppose we should have had eight fingers, or twelve!
>
> Some of the uncivilized people have used 20 as the base for their system of counting. Some South American Indians had the same word for twenty that they used for "one Indian" (twenty fingers and toes). An Indian with 21 horses would tell the number by saying, "One on the other Indian" (meaning as many as one Indian has fingers and toes, and one of another Indian's fingers). That our own ancestors often counted by twenties is shown by the various meanings of the word score in English and by similar words in other European languages.

As man became more civilized and language symbols began to represent syllables and sounds instead of words, the symbols for numbers became simpler.

* * * * * * * * *

The ancient Greeks (2500 years ago) used the letters of their alphabet for numbers. Various other symbols were used, and finally people came to use 1, 2, 3, 4, 5, 6, 7, 8, 9. These symbols are called *digits* or *figures* or *numerals.* For centuries after knowing and using these nine symbols people used the abacus for computation. You wonder why? Because it was convenient to have the rods determine the different values of the markers since there was no way to write the digits so that anyone else would know whether one stood for tens, hundreds, or units. They did not know how to write 305 as we do because the symbol 0, which we call *zero* or *cipher* or *naught,* was unknown. On the abacus if no marker of value ten was to be used, the upper end of the rod was simply left empty. To write it is 3 5 would confuse it with thirty-five. The Hindus (in Asia) about 1400 years ago conceived the idea of using a symbol to mean "nothing" and completed the system of numbers as we use them. The Arabs learned this method from the Hindus and passed it on to our European ancestors. Hence we call it the *Hindu-Arabic system.*

When fractions are taken up, why not let the pupil gain some such information as this:

Long, long ago in the days before historical records were kept, indeed before men had learned to write even with picture symbols, when people could count only by the help of fingers and toes, they had no need for fractions. Their herds could be counted in whole numbers. Their war clubs could be counted in whole numbers. Such an idea as half a tent or half a tree, did not occur to them. As they became more civilized, tilled the fields and raised grain, or in various ways dealt with smaller things, the idea of measuring things gradually entered their minds. Corn and wheat came to be measured by bushels (or by some similar measure) and soon they had half-bushels and other fractional parts. It was thus that the idea of a fraction probably originated. The invention and use of fractions marked a very important step forward in civilization.

Let us imagine a schoolboy of ancient Babylonia in southern Asia as he studied arithmetic more than 6000 years ago. With a hard, blunt-pointed instrument he made peculiar-looking wedge-shaped symbols on a soft, smooth, clay surface. He could erase them with a flat board by patting the surface smooth again. His textbook (if he had any) had been written the same way on a sort of clay brick and had been baked hard so that the symbols could not be erased. You wonder whether he had to learn fractions? He did, and to you his fractions may seem clumsy. All his fractions were 60ths, 3600ths, 216000ths, and so on, all the denominators being multiples of 60, just as our decimals are based on 10ths, 100ths, 1000ths, and so on, all the denominators being multiples of 10. He wrote them much as you write decimals, omitting the denominator.

About the same time that the Babylonian youth was dealing with his 60ths, some schoolboy down in Egypt was struggling to express all his fractions with ones for numerators. He was taught to write 2/3 as (½ + 1/6), 7/8 as (½ + ¼ + ⅛).

A few thousand years later we find a Roman schoolboy basing his fractions on 12ths, or with multiples of 12 as denominators, as the Babylonian used 60ths. We deal with problems in business and in common life that would have been very dificult, or quite imposible, for even the most learned scholars of those ancient times.

With such glimpses into the history of numbers the student will gain an appreciation of the fact that the processes of our number system were not handed down to us from a mountain of inspiration but are the fruits of laborious thinking in the long and tedious journey from barbarism to civilization and were developed as necessity demanded them. He should be made to see what a heritage is his.

81. The Most Important and Most Practical Thing in Arithmetic. Something has been said about eliminating from the course in junior high school mathematics certain obsolete and useless parts of arithmetic and enriching the course with the new subject matter. The cry for practical education has also forced the elimination from the course of many parts of arithmetic that deal with highly technical or theoretical commercial usage. In the zealous work of elimination some educators have decreased too much the work on the fundamental operations on numbers. In discussing what to leave out this question is naturally suggested: What is the most important and most practical thing in arithmetic? The answer is unmistakable: **The most important and most practical thing to be learned in arithmetic is a clear understanding and efficient use of the fundamental operations of addition, subtraction, multiplication, and division on integers, common fractions, and decimals.** These are the parts of arithmetic used daily by the average person and used more than anything else learned in the whole school curriculum except reading and writing. Any course is deficient that neglects to emphasize these things and fails to provide sufficient material for drill on these operations.

Authors of texts have naturally striven to meet the demands of the so-called newer education. In many texts the omission of the fundamentals and the crowding in of irrelevant material is evident. In the seventh grade text of a junior high school series published in the last decade only thirty-eight pages are devoted to the treatment of the fundamental operations on integers, fractions and decimals and problems involving especially these operations. In the same text fifty-five pages are given over to the treatment of new subjects like equations, graphs, and intuitive geometry. The remaining forty-five pages treat the usual applications of arithmetic. This deficiency in purely arithmetic computation was not made up in the eighth and ninth grade texts of the series. Twenty-five per cent of the entire series was made up of intuitive and demonstrative geometry and about forty per cent of the material of the three-year course was devoted to arithmetic and its applications. This illustrates the extreme to which some authors have gone in trying to break away from the old order of things.

If a teacher must use a text in which there is not enough material for the development of a reasonable amount of skill in the use of the fundamental operations of arithmetic he should supplement the text. If, on the other hand, an out-of-date text must be used, one which emphasizes on page after page the little used material of arithmetic and omits any reference to equations, graphs, and intuitive geometry, the teacher should provide himself with some good supplementary series of texts on junior high school mathmatics and use such of the new subject matter as he can in the place of certain material in his text.

References

Clark, Mathematics in the Junior High School.

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CHAPTER X

SUBJECT MATTER AND OBJECTIVES OF ELEMENTARY ALGEBRA

82. The Subject Matter. As in arithmetic, so in algebra has it been necessary to omit some of the traditional material which is a relic of the days when algebra was studied mainly as an end in itself; that is, in order that pupil might become proficient in algebraic technique. The National Committee recommends (Report, page 27) the omission of excessive drill in complicated algebraic technique and for all courses below the tenth grade the omission of the following subjects:

> Highest common factor and lowest common multiple, except the simplest cases involved in the addition of simple fractions.
>
> The theorems on proportion relating to alternation, inversion, composition and division.
>
> Literal equations, except such as appear in common formulas, including the derivation of formulas and geometric relations, or to show how needless computation may be avoided.
>
> Radicals, except as indicated in a previous section.
>
> (Note: The radicals indicated in the previous section are simple cases of monomial radicals.)
>
> Square root of polynomials.
>
> Cube root.
>
> Theory of exponents.
>
> Simultaneous equations in more than two unknowns.
>
> The binomial theorem.
>
> Imaginary and complex numbers.
>
> Radical equations except such as arise in dealing with elementary formulas.

The material required for the first year of algebra by the College Entrance Examination Board in its latest report (May 15, 1923) is a good criterion for the content of the course. The requirements of the Board, except for minor details, agree with the recommendations of the National Committee. In all examinations set by the board for the first unit of credit in elementary algebra, the following topics are to be covered:

> 1. The meaning, use, evaluation, and necessary transformations of simple formulas involving ideas with which the pupil is familiar, and the derivation of such formulas from rules expressed in words.
>
> 2. The graph, and graphical representation in general. The construction and interpretation of graphs.

3. Negative numbers; their meaning and use.
4. Linear equations in one unknown quantity, and simultaneous linear equations involving two unknown quantities, with verification of results. Problems.
5. Ratio, as a case of simple fractions; proportion, as a case of an equation between two ratios; variation. Problems.
6. The essentials of algebraic technique, including:
    - a) The four fundamental operations.
    - b) Factoring of the following types:
        - 1) Monomial factors;
        - 2) The difference of two squares;
        - 3) Trinomials of the type $x^2 + px + q$.
    - c) Fractions, including complex fractions of simple type.
    - d) Numerical verifications of the results secured under a), b), and c).
7. Exponents and radicals.
8. Numerical Trigonometry.

    The use of the sine, cosine, and tangent in solving right triangles.

    The use of four-place tables of natural trigonometric functions is assumed, but the teacher may find it useful to include some preliminary work with three-place tables.

    It is important that the pupil should acquire facility in simple interpolation; in general, emphasis should be laid on carrying the computation to the limit of accuracy permitted by the table.

Detailed requirements in exponents and radicals limit radicals to simple monomial types. No process for finding the square root of a polynomial is required.

For the third semester of algebra the required subjects are as follows:

1) Numerical and literal quadratic equations in one unknown quantity. Problems.

2) The binomial theorem for positive integral exponents, with applications.

3) Arithmetic and geometric series.

4) Simultaneous linear equations in three unknown quantities.

5) Simultaneous equations, consisting of one quadratic and one linear equation, or of two quadratic equations of the following types:

$$ax^2 + by^2 = c \qquad a_1x^2 + b_1y^2 = c_1$$
$$xy = h \qquad a_2x^2 + b_2y^2 = c_2$$

6) Exponents and radicals.

- a) The theory and use of fractional, negative, and zero exponents.
- b) The rationalization of the denominator in cases including binomial surd denominators.
- c) The solution of irrational equations.

7) Logarithms.

- a) The fundamental formulas;
- b) Computation by four-place tables;
- c) Applications to the trigonometry of the right triangle.

The detailed requirements under the first topic include the solution of the general quadratic and the interpretation of the graph of the quadratic function. Every teacher should write to the College Entrance Examination Board, 431 West 117th Street, New York City for a copy of its report and study the details of the requirements as well as the comments of the Board. One copy of the report will be sent free to every high school teacher asking for it.

There can be no authority more trustworthy to which a teacher may look for guidance in determining the content of the course in algebra than either the College Entrance Examination Board or the National Committee. These two groups agree in all essentials of the course. Whether the first year of algebra is taught in the ninth grade as a separate course or as parts of a three-year junior high school curriculum makes no difference. The subject matter remains the same. What is said in this chapter and the next applies to algebra wherever and however taught.

83. A Fourfold Objective. The word "objective" as used in the title does not mean that a teacher has a plan to cover so many pages, or certain chapters, in one semester or year. It means that a certain mental growth, a certain knowledge of facts, a certain ability to use the attained knowledge to a fair degree of proficiency in ordinary life is expected of the class at the end of the course whether or not a given number of topics has been covered or certain types of problems studied. It should be the aim of every teacher of algebra:

(1) To secure as clear a fundamental understanding of the reasons for each mechanical process as possible.

(2) To develop a reasonable degree of efficiency and accuracy in mechanical operations.

(3) To develop ability to think independently in the symbolic representation of facts and conditions.

(4) To make his students see the usefulness of the subject and its vital relation to the development of civilization.

To slight the first objective mentioned is to encourage memorization of processes poorly understood. This fails to develop the habit of independent thinking so essential in mathematics and so useful in life. It also makes it harder for the teacher to attain the second objective, for an understanding of the process, the ability to see through the whole operation understandingly, is one of the best means of detecting mistakes. For example, the reason for changing the signs of all terms in parentheses preceded by the minus sign is that the minus sign indicates subtraction and the rule for subtraction is that the subtrahend with its sign changed is added to the minuend. The pupil should see that when he removes the minus sign and the parentheses he is merely changing the time arrangement of his subtraction process; that is, he is first changing all the signs preparatory to addition of all the terms later when he is collecting them. If he thinks "subtraction" every time he sees the minus sign before parentheses, he is less likely to forget to change some or all of the signs. Again, in learning that the product of the sum and the difference of two numbers is the difference of their squares the pupil should be kept conscious of the fact that multiplication like $(a + b)(c - d)$ has been performed and because of the symmetry in the occurence of the terms with their positive and negative signs the only remaining terms of the four that would be expected in the product are the products of the two first and two last terms of the factors. If he learns "by rote" that $(a + b)(a - b) = a^2 - b^2$ he sometimes begins to think it was obtained as the sum of the products $a \cdot a + b\ (-b)$ and then writes $(a + b)(a + b) = a^2 + b^2$. The mistake is the result of too little emphasis on the first aim stated above. It is not meant that these things should merely be brought to the attention of the pupil when he first meets the rule, but that it should be kept in his mind until he carries the idea subconsciously as he works. Another example is that of transposition in equations. Some of the best teachers do not allow it as a process in the first year of algebra, but insist on the pupil adding to or subtracting from both sides of the equation all through the course. It is certainly true that such emphasis on that fundamental principle would help to avoid many of the common mistakes in the solution of equations. For example, consider the familiar error in the solution of quadratic equations where the third term of the incomplete trinomial square is added only to the first member of the equation. **A sub-conscious knowledge of the underlying principles of a mechanical operation gives the student a command of the situation that is invaluable.** Just how far this fundamental understanding may go depends on both pupil and teacher.

The word reasonable is used advisedly in the statement of the second objective. We can not expect to make rapid calculators in algebra out of children to whom the subject is new and strange. Nevertheless, our teaching of mathematics fails in one of its missions if the pupils are not made to appreciate the value of correctness, exactness, precision and efficiency. Some of them may continue to study some branch of science. No teacher of science can endure inefficient manipulation of the processes of mechanical algebra without complaint. **It is of little value to know how to solve a problem if one's mechanical computation is so faulty that his answer is seldom correct.**

The third objective is accomplished through teaching the subjects that require a minimum of mechanical manipulation and a maximum of independent thinking. Teachers are so prone to omit graphs and neglect the emphasis on the functional relation because they do not understand the importance of the former (or do not understand the subject itself) and do not know just what the latter means. The remainder of this chapter will be entirely devoted to a consideration of the latter subject. It is also easy to omit much of the portion devoted to verbal problems because in the rush to cover certain pages something must be slighted and the pupils' ability to continue (the mechanical processes) in the next course will be hampered least by the omission of some of the verbal problems.

Teaching a pupil to think out the solution of verbal problems is hard because he has probably never had sufficient training in independent thinking. For that reason the emphasis should be stronger on this part of the work. To neglect it is to rob him of his main opportunity in mathematics, and weaken his ability to apply what he has learned. This third objective is very vitally connected with the teaching of the practical applications of the subject. No pupil can be given much worth while drill in practical applications by merely working certain specific, so-called applied problems, which probably deal with things entirely outside his experience and usually apart from his interests. His best training along that line is to have developed a habit of independent thinking and an ability to interpret everyday conditions symbolically. Practice on a few types of applied problems without ability to think for himself means that he might solve that particular kind of problems if he meets it and remembers how he did it in school. On the other hand, the development of the habit of examining a problem for all detailed facts and immediately marshaling those facts (in whatever form they be given) in brief symbolic statement will serve a student well in analyzing any kind of problem he meets and applying, if possible, the principles of algebra. **Aim to develop independent thinkers, with initiative and resourcefulness.**

The fourth objective mentioned has been discussed at length in preceding chapters in connection with mathematics in general. It is not restricted to algebra. It is attained through the use of applied problems. By applied problems we mean problems that use the principles of algebra in the everyday life of the pupil or in familiar situations relating to commerce and industry. Suppose, for example, that the class is studying evaluation of formulas and the students need drill in the mechanical process of substituting numbers for literal symbols. Instead of the too familiar type of literal expressions involving letters like a, b, c, m, n, etc., with values arbitrarily assigned in expressions arbitrarily made up, suppose we give them the formula for the horse power of an automobile engine; namely, $h. p. = d^2 n/2.5$, where d is the diameter of the cylinders in inches and n the number of cylinders. The diameters of the cylinders of all the familiar makes of automobiles can easily be secured. It is easy to see how the natural interest of the class in comparing horse power will motivate the needed drill. "Made up" problems about the heights of mountains, lengths of rivers, etc., contribute little or nothing to this objective. They have their value in providing conditions for training in the use of symbolic algebra.

84. The Importance of the Functional Relation in Connection with the Objectives of Algebra.

Definitions and Illustrations

In §76 a statement was quoted from the Report of the National Committee concerning the functional relation. One sentence will be repeated here, as follows: "The one great idea which is best adapted to unify the course is the functional relation". Let us have at the outset of this discussion a clear understanding of what is meant in mathematics by the expression "function of a variable".

The word function inplies variable. A function is a variable. To understand the term variable let us first consider the definition of constant.

A **constant quantity** is one whose value never changes. Some examples are: (1) Any one of the numbers of arithmetic, say, 3, 4, 17, ... (2) a geometric constant, π (= 3.1416, nearly) the ratio of the circumference of any circle to its diameter, which we would consider a rather remarkable and surprising relation were we not so familiar with it; (3) the force of gravity (at sea level), called g (= 32.2, nearly), which may be called a constant in nature. A study of the sciences reveals many more situations in which some quantity never changes value. Constants exist all about us beyond the power of our influence.

A **variable** quantity is one whose value changes. For example, the length of the day (daylight), the price of cotton, the radii of different circles, intervals of time, the annual rainfall at a given place, etc.

We say that a function of a variable is a quantity which depends on the variable for its value. This is true but it is not the best expression for the definition of a function. In more precise language, if two variables are so related that for any value assigned to one of them one or more definite values are determined for the other, the second variable is said to be a **function** of the first. The interdependence between the variables is the most noticeable relation expressed by the definition of function. We may find illustrations of the functional relation everywhere. The circumference of a circle depends on the length of its radius; the distance traversed by a body falling from rest toward the center of the earth depends on the length of time it has been falling; the annual receipts at any postoffice depend on the population of the town and surrounding country served by the postoffice; the enrollment of the public schools of any city depends on the population of the city; the number of overcoats sold annually in any locality depends on the number of degrees registered by the thermometers of that locality; the number of bushels of wheat per acre depends on the number of inches of rainfall, the number of insect enemies, and other things; the price of cotton depends also on the rainfall, the temperature, the boll weevil, and other things.

The mathematical expression of the first two functions mentioned in the preceding paragraph furnish excellent illustrations of the meaning of constant, variable, and function. Consider the relation $C = 2\pi R$. 2 and π are constants, R is a variable and, as written, C is a function of R. The expression $R = C/2\pi$ shows R as a function of C. Such interdependence exists in all functions. Again, $s = gt^2/2$ expresses the law of falling bodies (under constant acceleration in a vacuum) where t is the number of seconds elapsed in the time of flight and s the number of feet traversed in t seconds. Here g and 2 are constants, t is a variable and s a function of t. Some functional relations are thus simply expressed by a mathematical equation. Some are not so simply expressed; for example, the other illustration above, depending on several variables simultaneously. Some have never been expressed in mathematical form other than being pictured by a statistical graph, for they involve too many variable conditions of such irregularity. They are functional relations, nevertheless. Upon man's ability to discover and express mathematically many functional relations of nature and geometry has hinged the development of much of the science of modern engineering and many other branches of applied science. Because of the ever present functional relations (some expressible and some not) it is pertinent to consider the subject in connection with a course in high school mathematics. Let it be understood here and now that the formal definition of function is not necessarily considered in the high school course, the name probably is not mentioned in first year algebra.

Examples of Functional Relations

The contents of this article must be considered only as supplementary to Chapter VII, "The Function Concept in Secondary Mathematics", of the Report of the National Committee. In this report the committee considers examples of functional relationships in algebra, geometry and trigonometry. Six illustrations are there given of portions of the subject matter of algebra that lend themselves readily as fields for teaching the functional relation. They are as follows:

(1) Use of letters for numbers.
(2) Equations.
(3) Formulas of pure science and of practical affairs.
(4) Formulas of pure algebra.
(5) Tables.
(6) Graphs.

Every teacher should secure a copy of this report, if only to borrow it from a friend or some library. I shall include here not even a brief of the discussion of the above topics by the Committee but will give some illustrations from my own experience where the functional relation was being emphasized unconsciously.

My class was studying "Shop Problems in Mathematics". Some problems were like this: If a cylindrical measure 10 inches deep holds a bushel, what is its diameter? What is the diameter of a half-bushel measure of the same depth? Of a peck measure of the same depth? After answering the first question by solving $V = \pi r^2 h$ for r when $V = 2150.4$ and $h = 10$, the class of boys was practically a unit in declaring that the diameter of the half-bushel measure would be half the diameter of the bushel measure of the same depth, and the diameter of the peck measure one-fourth the diameter of the bushel. I showed them in what way V changed as r changed and, conversely, how r changed as V changed and corrected their erroneous notions. This was a simple consideration of how r depends on V, or in what sense r is a function of V.

Another type of problem was this: If two sewer pipes, each one foot in diameter, converge, how large must the single pipe be to carry away the sewage without overflow when the pipes are running full? Reversing the idea in a similar problem, how many three-quarter-inch house-service pipes will a four-inch water main supply without decreasing the pressure? These involve simple functional relations in geometry; namely, the area of a circle varies directly as the square of the diameter. Many a teacher in problems like these and many other problems has been emphasizing the functional relation unconsciously, and while doing it some have been fighting the introduction of what they thought was the functional relation into elementary mathematics.

In teaching the solution of the quadratic equation by means of the formula, it seems to me that a teacher could not help but emphasize the functional relation (though perhaps not calling it by name) in showing how the values of the roots depend on the coefficients a, b, and c. It is a familiar fact expressed in higher algebra that the roots of an algebraic equation are functions of the coefficients. Another place where the functional relation occurs in quadratic equations is in the term added to both members to complete the square. This term is a function of the coefficient of the term of the first degree; namely, the square of half this coefficient. Any good teacher can multiply indefinitely the number of illustrations like the foregoing.

Why Emphasize the Functional Relation?

This question at once arises in the minds of some: If it is to be understood, as mentioned before, that the term "function" or "functional relation" may not be used with a class in first year algebra, wherein lies the value of the teacher's constant consciousness of this hidden idea and the suggested emphasis by the teacher on the mysterious relation? Let us consider a specific problem in answer. Given $V = \pi r^2 h$, find V when $r = 7$, $h = 10$ ($\pi = 22/7$). This type of problem occurs early in beginning algebra. It may be, and usually is, used simply as a numerical valuation of literal expressions, which requires little or no independent thought, but involves merely the mechanical processes of substitution and multiplication. It may be made a live problem in speculative thought by a consideration of the changes produced in V by changing r while h remains fixed, or by changing h while r means fixed. Any student will discover that the nature of the change produced in V by a change in r differs from the change produced in V by a similar change in h and some students may be encouraged to discover the nature of the difference, expressing it quantitatively. The substitution of various values for r and h in making up a table of values for the purpose of the investigation furnishes just as good a drill in evaluation of formulas as the evaluation of such commonplaces as ab^2, $3a^3b-b^2$, etc., when a and b are given. Moreover, it will add to the mechanical drill the element of independent thinking so lacking in much of algebra and so vital in the study of mathematics. It will at the same time furnish a motive for computing values in that the goal is to compare these computed values of V. The illustration of the formula for the horsepower of an automobile engine (§83) could also be repeated here. This drill in independent thinking is an excellent preparation for solution of verbal problems where relations between quantities must be carefully considered and correctly expressed.

If we are to reach that goal of mathematics teaching, the development of independent and resourceful thinkers, we lose a very great help to that end when we slight the consideration of the functional relations existing in elementary algebra. Furthermore, the function concept is so woven into the body of mathematics that we hinder the future growth of such of our pupils as may proceed further with the subject by neglecting it in early courses.

The functional relation may or may not be the one unifying idea around which the course in algebra should be built. There seem to be other concepts in algebra quite as vital. There is no doubt of its far reaching influence in mathematics and in the practical experience of life. There is no doubt that it can serve the purpose of encouraging independent thought and for such a use alone it is worth emphasizing.

References

The Reorganization of Mathematics in Secondary Education, (Report of the National Committee) Chap. VII.

J. W. A. Young, The Teaching of Mathematics.

Schultze, The Teaching of Mathematics.

CHAPTER XI

SUGGESTIONS ON TEACHING SOME CRITICAL PARTS OF ALGEBRA

85. **The Introduction to Algebra.** A great deal depends on a student's first impressions and conceptions of a new study. In the introduction his interest must be held, he must not be plunged in over his depth, and there must be a point of contact. While the textbook largely determines the subject matter used in the beginning, a teacher may have some liberty in the rearrangement of the subject matter and can make a good introduction even if compelled to use a poor text. The three traditional avenues of approach to the subject are: (1) evaluation of formulas; (2) solution of simple problems by means of equations; and (3) negative numbers. The third subject mentioned may be disposed of with the remark that it can be made to fulfill only one of the requirements mentioned above; namely, his interest may be held. It is losing caste as a method of introduction to algebra. Recent texts postpone the introduction of negative numbers until one-fourth to one-third of the book is completed.

The strongest point in favor of the use of the evaluation of formulas as an introduction to algebra is that it furnishes a point of contact. Such rules for mensuration in arithmetic as the area of a rectangular field, a triangle, or a circle, as well as the rules for percentage and interest, can be expressd by appropriate symbols, and the average student has had some experience in so expressing them. The formula $A = \pi R^2$ is probably familiar to the whole class, and perhaps also $P = B \times R$. From the use of these the pupil can be led to see **the general principle of using literal symbols for numbers** which is one of the **new concepts** he gets in the study of algebra. With only a few principles of operation to learn he is not likely to experience serious difficulty with this type of work. His interest will depend on whether he uses arbitrarily made up expressions like $ab^3 + 3a^2b$ or formulas expressing relations within his experience. Read again the illustration used in discussing the fourth objective in the teaching of algebra (last paragraph of §83). The following extracts from a recent text (first chapter) also illustrate the point.

> 1. *Introduction.* The construction and the use of formulas are illustrated by the mensuration formulas that you have had in previous courses. Consider, for example, the rule for the volume of a rectangular block; namely, that the

volume is equal to the product of the length, width, and height. This means, briefly, that

$$\text{volume} = \text{length times width times height.}$$

or, more briefly,

$$V = l \times w \times h.$$

We use this formula as follows.

Example. How many cubic yards of earth must be removed in digging a basement 24 feet long, 18 feet wide, and 4½ feet deep?

In this problem, the dimensions being reduced to yards, $l = 8$, $w = 6$, $h = 1.5$; hence

$$V = 8 \times 6 \times 1.5 = 72. \quad \textit{Ans. } 72 \text{ cu. yd.}$$

A *formula* is an expression of some rule or law in abbreviated form by the use of letters as symbols for numbers. The use of formulas avoids long rules and tedious explanations and simplifies the solution of problems.

To further simplify and abbreviate the writing of formulas we omit the symbol × for multiplication, or use in its place a dot. Hence

(a) $l \times w \times h$ is written lwh or $l \cdot w \cdot h$;

(b) $4ab$ means 4 times a times b.

EXERCISES

1. *The perimeter* of a plane figure is the total length of its boundary. If the figure is bounded by straight lines (called *sides*), the perimeter is the sum of the sides. Let l = the length, w = the width, and p = the perimeter of a rectangle.

(a) What does the formula $p = l + w + l + w$ mean?

(b) Express in words the formula $p = 2l + 2w$.

(c) Find the perimeter of a rectangle 12″ × 16″. Substitute the given values for the corresponding letters in formula *(b)*. Is this formula better than formula *(a)*? Why?

(d) The foundation for a house is in the shape of a rectangle 26′ × 28′, outside measurement. What is the total length of this foundation wall?

(e) How many rods of fencing are required for a rectangular field 48 rods wide and 86 rods long?

* * * * * * * * *

2. *The Percentage Formula.* All computations of percentage involve the three quantities, pecentage, rate, and base, where

b = the base (the number on which the percentage is found),

r = the rate (the number of hundredths to be taken),

p = the percentage (the number found by taking a certain per cent of the base).

EXERCISES

1. Express in words the formula $p = rb$.

2. In a school which had an enrollment of 240 pupils, 25 per cent of them were under weight. How many pupils were under weight? (Use the formula of Exercise 1.)

3. Mr. Brown owns a farm 160 acres. 70% of it is under cultivation, 20% of it is pasture land, 8% of it is timber, and the remaining 2% is used for buildings, etc. How many acres are used for each purpose?

4. A school which enrolled 1260 pupils in 1922 reported a 16 2/3% increase in 1923. How many more pupils were enrolled in 1923 than in 1922?

5. In buying and selling, the gain or loss is computed as a per cent of the cost or of the selling price. Let $c =$ the cost, $s =$ the selling price, $g =$ the gain, $l =$ the loss, and $r =$ the rate (per cent). What is the meaning of $g = rc$? $l = rc$? $g = rs$?

6. A man bought a house for $4500 and sold it at a gain of 20% on the cost. How much did he gain? (Substitute the words gain, cost, and rate in the formula $g = rc$.)

7. A house was sold for $5400, which was a gain, or profit, 16 2/3% on the selling price. What was the profit?

. . . .

3. The Distance-Rate-Time Formula.

The following exercises give practice in making and using a new formula. Solve each exercise and explain your solution.

EXERCISES

1. A boy walks at the average rate of 3½ miles an hour. How far does he go in 3 hours? In x hours?

2. A train travels at an average rate of 40 miles per hour. How far does it go in 2½ hours? in $3x/4$ hours?

3. A steamship moves at a uniform rate of 22½ miles an hour. How far does it go in 4 hours? In y hours?

4. A boy runs at an average rate of 6 miles an hour. How far does he go in 2 hours and 15 minutes? In $x/4$ hours?

5. An airplane is traveling at a uniform rate of 128 miles an hour. How far does it go in 45 minutes? In y minutes?

6. In each of Exercises 1-5 there are three quantities that have a certain relation to each other. The quantities are *distance, rate, and time.* Using the letters d, r, and t, write a formula to show their relation. Explain what each letter stands for.

. . . .

4. Mathematical Symbols. You have seen in the formulas of the preceding sections the advantage of an abbreviated notation. For at least three thousand years after men began to develop rules for the mathematical processes, they worked without such an abbreviated notation as we have to-day. As civilization advanced and the problems of living and trading became more complicated, the *need* of abbreviated symbols forced their use.

(Two paragraphs of history of the development of symbolic algebra are omitted here.)

The completely abbreviated mathematical notation, as you will learn it in this course, has been in use less than 300 years, and represents the best that inventive minds have been able to devise. It is valuable and practical, and you should try to become proficient in its use. The various principles governing the use of mathematical symbols will be taught in the course as needed. The following exercises will illustrate some of the more elementary principles.

Another method that finds popular favor is the use of simple equations of the first degree at the beginning of algebra to facilitate the solutions of problems that are difficult to analyze by arithmetic methods. This use of the equation is **another** of the **new concepts** the student gets in algebra. It also involves the use of a literal symbol to represent a number. It will hold his interest and furnish a point of contact if the problems used are of the type he has solved in arithmetic, as they should be. The process of solving equations is more complex than the process of evaluation of formulas. Both processes give a pupil an insight into the applications of algebra at the beginning and the former ought to inspire in him a respect for its power as a problem solver. The best method is probably to begin with the evaluation of formulas and pass to the solution of equations when the interest of the class passes its high point and begins to wane. **Let not the amount of material in the text but the progress and interest of the majority of the class determine the time spent on each.** It seems to me that these two new concepts in mathematics are enough to give the student in the first month. The notion of **negative numbers,** which is another great **new concept** that belongs distinctively to algebra, should be reserved until he has feet on "solid rock".

86. Treatment of Negative Numbers. The following four rules should be observed:

(1) Make clear the notion of an arbitrary selection of a symbol to represent things **opposite** in meaning to the numbers of arithmetic.

(2) Put strong emphasis on the ideas of **oppositeness, position,** and **direction.**

(3) Make it clear that the rules of operation arise from arbitrary assumptions, but are such that the results are to be reasonably expected if our computation is to harmonize with that of arithmetic.

(4) By the use of numerous illustrations of the applications of negative numbers make the pupils feel that these numbers are needed in our actual experience.

It is interesting to note the change of attitude toward the presentation of the law of signs in multiplication. In a text on elementary algebra used in the latter half of the nineteenth century the author "demonstrates" the commutative law of multiplication and then the law of signs. He assumes "a sash containing **a** vertical rows and **b** horizontal rows of panes". By taking "**a** vertical rows and **b** panes in each row" he gets the number of panes to be **ab**. By taking "**b** horizontal rows and **a** panes in each row" he gets **ba** as the number of panes. "Hence", he concludes, "it follows, that the product of two factors is the same whichever be made the multiplier". A few pages further he derives the law of signs for negative numbers. He shows that $(-\mathbf{a})\,\mathbf{b} = -\mathbf{ab}$ and and that this is the same as $\mathbf{b}\,(-\mathbf{a})$, "since", says he, "when two quantities are to be multiplied together, either may be made the multiplier" (and here he refers to his "proof" that $\mathbf{ab} = \mathbf{ba}$, which was made on the basis of **a** and **b** representing "rows" and "panes in a row"). I have had alert freshman students in high school who, if faced with that reference to the "panes in a row" as an authority in such a case, would undoubtedly ask, "How could he have $-\mathbf{a}$ panes in any row?" The logical procedure in this case has already been discussed (see §§25, 35). In each of two modern texts in algebra examined the author says in explaining the same situation to make it clear (but not claiming that he is making a demonstration of the fact), "since in arithmetic we have $\mathbf{ab} = \mathbf{ba}$, we will **assume** that $(-\mathbf{a})\,\mathbf{b} = \mathbf{b}\,(-\mathbf{a})$". This is a statement of fact in a situation where a logical development can not be used.

The following extracts from a recent text illustrate a modern method (and mathematical) of dealing with the law of signs in multiplication. Instead of **proofs** we have **examples** of how it is actually done, followed by an argument showing that by the nature of the negative numbers ("opposite in quality to positive numbers") we should not be surprised to find the rules as they are.

44. Laws of Signs in Multiplication.

Example 1. Find the product of $+4$ and $+3$.

Solution. Consider $+3$ the multiplier, then

$(+3) \times (+4) = +12$, for this means

$(+4) + (+4) + (+4)$; that is, $+4$ is to be added three times.

Ans. $+12$.

Example 2. Find the product of -4 and $+3$.

Solution. Consider $+3$ the multiplier, then

$(+3) \times (-4) = -12$, for this means

$(-4) + (-4) + (-4)$; that is, that -4 is to be added three times.

Ans. -12.

Example 3. Find the product of $+4$ and -3.

Solution. Consider -3 the multiplier; as this multiplier is the negative of the multiplier in Example 2, this product is the negative of that product; that is,

$$(-3) \times (+4) = -12.$$

(Compare this answer with that of Example 1.) *Ans.* -12.

Example 4. Find the product of -4 and -3.

Solution. Consider -3 the multiplier; as the multiplier is the negative of the multiplier in Example 2, this product is the negative of that product; that is,

$$(-3) \times (-4) = +12.$$

(Compare this answer with that of Example 2.). *Ans.* $+12$

We may state these principles of operation with the signed numbers as follows:

A positive number times a *positive* number gives a ***positive product.*** (Example 1.)

A positive number times a *negative* number gives a ***negative product.*** (Example 2.)

A negative number times a *positive* number gives a ***negative product.*** (Example 3.)

A negative number times a *negative* number gives a ***positive product.*** (Example 4.)

From Examples 3 and 4 it is seen that when the multiplier is negative the sign of the product is opposite to that of the multiplicand. Compare these results with those of Examples 1 and 2 where a positive multiplier produces a product whose sign is the same as the sign of the multiplicand and notice that this is in accord with the definition of negative number as "opposite in quality" to that of a positive number. One of the results of multiplying by a negative number is to change the sign. In particular, if a number is multiplied by — 1, the only effect is to change the sign.

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Because of your unfamiliarity with negative numbers these results may not seem so natural to you as do the operations with the numbers of arithmetic to which you have been long accustomed. A study of the four examples will convince you that the results do harmonize with the definition and interpretation of negative numbers. Accept them as stated and become accustomed to them through use.

Many and varied are the ways of illustrating the fact that the results obtained by the use of the law of signs should naturally be expected. The following[56] illustration uses the law of the lever: Since negative numbers are defined as numbers used to express values **opposite** in meaning to the corresponding positive numbers, let us call the right arm of the lever positive and the left arm negative. Also, a force acting upward will be called positive and one acting downward will be called negative. If a force applied causes the lever to turn counter-clockwise let us call the effect positive; while if it turns clockwise we shall call it negative. Thus, by using sets of familiar numbers, the authors show that a positive force acting on the right side and a negative force acting on the left side produce the same results; namely, positive; while a negative (down) force on the right (positive) side will cause a negative rotation, as will a positive force on the negative side. The rotation, or turning, is by the law of the lever the product of the force and the lever arm. In another text[57] is suggested a laboratory exercise with a simply made balance that brings out the same idea.

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[56] Keal and Phelps, *Secondary Mathematics,* I, p. 51, Atkinson, Mentzer, & Co., New York (1917).

[57] Schorling and Reeve, *General Mathematics,* p. 180. Ginn, (1919).

In yet another text the authors[58] consider future time as positive and past as negative, saving money as a positive act and wasting it as negative, and show that saving a certain amount each month for five months (+) (+) increases your wealth (+) and if you have wasted the same amount monthly for five months (−) (−) your assets had the same increased value five months ago that they would have five months hence on the (+) (+) operation. An analogous argument is used for unlike signs.

In all these attempts to show the student that the new laws are to be expected one idea predominates; namely, the idea of **oppositeness** of the negative number. A teacher should use any good device that comes to mind which he thinks would appeal best to his own students rather than borrow one because it is published in a book. If a teacher does not do some independent thinking the students are not likely to get the habit.

With the ideas of oppositeness, position, and direction uppermost it seems to me that a student needs no especially prepared illustration to see that adding a positive and subtracting (opposite of adding) a negative (opposite of positive) should produce the same result. It amounts to a reversal of a reversed direction. This is exactly according to the proof of the fact as brought out in §35.

87. Equations.

General Considerations

On this topic, one of the very oldest in algebra, it is hard to say anything that experienced teachers do not know. With the hope of helping some beginner in the ranks of teachers of mathematics, the following suggestions are offered.

That the idea of a balance is the best point of contact in teaching equations at the beginning goes without question. Practically all modern textbooks make it easy for a teacher to use the idea by including such illustrations, usually accompanied by drawings.

As the word opposite is the key in teaching negative numbers so is the expression "both members" (or both sides) the phrase to reiterate in teaching the equation. The students should master thoroughly the process of adding (or subtracting) the same quantity to (or from) both members of an equation before being introduced to the notion of transposition of terms from one member of an equation to another. I can testify from personal experience that in such equations as $13x - 4 = 6x + 7$ I am more likely to perform the mental subtraction of 6x from both sides than to transpose. A pupil can be trained to be conscious of such possible operations and perform them with a saving of time over the one who inevitably transposes without understanding why. A pupil should never be allowed to lose sight of the idea that **transposition is only a device for writing quickly the result of an addition or subtraction performed on both sides.** It should be added that it is an incomplete operation at best for the actual result must later be determined by the collection of terms. Perhaps it should not be discarded, since in long problems it is worth while as a short cut. It should not be taught before the second year of algebra.

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[58] Rugg and Clark, *Fundamentals of High School Mathematics*, p. 179, World Book Co., Chicago, (1919).

If the phrase "both members" has been kept sufficiently in the foreground in the study of equations it will minimize the number of mistakes like the following: (1) If $x/a + c/d = y$, some get by clearing of fractions, $dx + ac = y$. (2) In the equation $x^2 + 2ax = b$ some attempt to complete the square by writing $x^2 + 2ax + a^2 = b$.

There is no doubt in the minds of some that checking problems in algebra is sometimes overdone. Some suggested systems of checking are more laborious than the solution of the problems. Their values are not commensurate with the time and labor required to learn and use them. However, **checking the solutions of equations can never be overdone.** This is one place where checking should never be slighted.

In the solution of simultaneous linear equations too much prominence is given in the average text to the method of elimination by addition and subtraction. It is a device that can be used only on special types of equations in mathematics in general; namely, those that are homogenous in the unknowns as are the linear equations in two unknowns where it is usually introduced. On the other hand, the method of elimination by substitution is a standard method which follows a pupil through all mathematics and is usable more than any other method. Simultaneous equations in physics and other applied sciences are not always of the same degree and substitution is the most frequently used and the most reliable method of elimination. Elimination by substitution certainly ought to be emphasized much more than is being done by most teachers.

The Identical Equation

Unfortunately, two distinctly different definitions of identity, or identical equation, appear in the texts in algebra.

(1) An identical equation is one that is true (or satisfied) for any values of the unknown symbols involved.

(2) An identity is an equation with members exactly alike, or members that may be reduced to the same form.

From the logical standpoint either may be considered as a definition and the other proved as a theorem. From the standpoint of usefulness in mathematics (consider the problem of finding the undetermined coefficients in the subject of partial fractions) the first of these conceptions of an identical equation should be emphasized to the extent that it be given as the definition. In only five of the nine texts in elementary algebra recently examined is the definition so given. An examination of several standard complete treatises on algebra reveals the fact that all but one give the first definition. Inasmuch as one of the two concepts will be, as it has been, taught in elementary algebra it is certainly preferable to teach the one most valuable in future mathematics. The failure to emphasize the better concept (as is shown by information gleaned from my students in the past and from high school teachers) is an illustration of deficiencies pointed out in the first chapter.

To the pupil in beginning algebra the term conditional equation has a content only when presented in contrast to the idea of an identical equation being satisfied by any value of the unknown. The question of the equality of the members of a conditional equation is preceded by an if. The members are equal if the unknown has a certain value or values and the business, the whole business, of the mechanical solution of equations is to find the particular values that satisfy the condition.

As for the property mentioned in the second definition, the name itself suggests it without further emphasis. It is well to illustrate by concrete examples that both members can be made exactly alike. Whatever be the definition given in your text, let your pupils be so taught that the term "identical equation" means an equation whose property of equality does not depend in any way whatsoever on the value of any unknowns that appear.

Checking Irrational Equations

In a second course (third semester) in algebra the solution of radical (irrational) equations is considered. Given the equation, $-\sqrt{x} + \sqrt{x-9} = 1$. Squaring both sides and transposing the rational terms to the second member, we have $-2\sqrt{x^2-9x} = 10 - 2x$. If we square again and solve the resulting linear equation, we obtain $x = 25$. The reaction of the ordinary pupil in a problem like this is that since no involved steps were taken and since the result is a whole number there is no use checking the "solution". As a matter of fact, 25 will not satisfy the equation, indeed no number will. For reasons like this, if for no other, it is imperative that a student of algebra acquire the habit of checking all solutions of equations.

The discussion of extraneous roots and how they may be introduced in the progress of solution of an equation by (a) multiplying both members by a function of the unknown and (b) by raising both members to the same power can be found in detail in any standard college algebra. In the equation above, when we first squared both members of the equation we obtained the same result as if the second member had been -1 instead of $+1$. The equation $-\sqrt{x} + \sqrt{x-9} = -1$ is satisfied when $x = 25$. Or, we may see it this way: $(-\sqrt{x} + \sqrt{x-9})^2 = 1^2$ is the same as $(-\sqrt{x} + \sqrt{x-9})^2 - 1^2 = 0$, which may have the form, when factored, $(-\sqrt{x} + \sqrt{x-9} + 1)(-\sqrt{x} + \sqrt{x-9} - 1) = 0$. The first factor set equal to zero gives an equation that is satisfied when $x = 25$. The second factor can not become zero for any real value of x.

When teaching that both members of an equation may be multiplied by the same number without destroying the equality, it is only just to the pupil to emphasize the fact continually that the word number here means a fixed number (constant) and not a number like the unknown x. It may save him trouble later.

88. Fractions. In the study of fractions the processes of arithmetic are continued and extended to a wider application involving algebraic polynomials. The same pitfalls are to be avoided that we find in arithmetic. Most errors, aside from those due to that prolific source known as "a slip of the mind" arise from the failure of the pupil to keep subconsciously in his mind an understanding of the processes as mentioned in the discussion of the first aim in §83. It seems that too much stress can not be laid on such fundamental ideas as the following:

A fraction is simply an indicated quotient of a division that can not be completed in the sense of division in arithmetic.

A fraction is multiplied by multiplying the numerator (or dividing the denominator).

If both terms are multiplied by the same quantity the fraction's value is unchanged.

Many a student has passed through to college who is not sure that 2 times 2/3 is not 4/6. He should have the feeling instilled into his system that when both those terms are multiplied the value of the fraction is not changed. In such case he notices an error like that much as a musician "feels" a discordant note. The two following methods of teaching two different topics in fractions have been used with success.

In raising fractions to higher terms (or reducing to lower) keep before the mind of the pupil that both numerator and denominator are to be operated on. The fraction is "stepped up" (or down) to be in uniformity with the other fractions of the problem with respect to its denominator. Or, if a fractional radicand is to have its denominator rationalized, the fraction is "stepped up" to a form with a perfect-square denominator, (for example, $\sqrt{2/3} = \sqrt{6/9}$). In adding fractions the adherence to the old fashioned rule of "dividing the common denominator by the given denominator and multiplying the quotient by the given numerator for a new numerator" (if such be yet adhered to by any teacher) will drive out of the pupil's mind the proper relation of the old form of the fraction to the new form and upset his fundamental understanding of the processes as stated in the first objective in §83. For example, the fraction $3/(x-2)$ must have the denominator $x^2 + x - 6$. It is necessary to multiply $x - 2$ by $x + 3$ to obtain the denominator $x^2 + x - 6$, therefore we raise the fraction to higher terms by multiplying **both** the numerator and denominator by the necessary factor, $x + 3$.

In reducing complex fractions, whether the very simple types that may be introduced in first year algebra or the more difficult types that occur in the third semester of algebra, let us by all means discard the old method of simplifying the numerator by possibly performing an addition or subtraction and then inverting the terms of the denominator and multiplying by the numerator. Since a fraction may have both of its terms multiplied by the same factor without changing the value of the fraction, we multiply both the numerator and denominator by the least common denominator of the fractional forms occuring in the terms of the fraction and in one operation the complex fraction is made into a simple fraction. This method is exactly analogous to clearing of fractions in equations and the teaching of either one strengthens the pupil in the other.

For example, simplify:

$$\frac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{\dfrac{9}{x^2}-1}$$

If we multiply both numerator and denominator by x^3 we clear each of fractions. In one step we have

$$\frac{x^2-2x-3}{9x-x^3}$$

Compare this with the older method of treating the numerator and denominator as separate problems in addition of fractions and then inverting the terms of the denominator to multiply. The further reduction and simplification of this result involves the same steps by either method. As in the process of solving equations, the important word here is "both". Indeed with regard to the two operations of multiplication and division the two terms of a fraction bear a striking resemblance to the two members of an equation. But beware of carrying the analogy further; for example, to addition.

89. Verbal Problems. This is the most discouraging subject that a teacher encounters in algebra, mainly because it demands of the pupil resourcefulness rather than repetition, mental ingenuity more than memory and initiative instead of imitation. The importance of the subject was discussed in §83. A hard and fast rule for teaching verbal problems can no more be laid down than can the pupil be given an open sesame to the solution of all such problems. In this subject the teacher must have some resourcefulness, ingenuity and initiative as well as the pupil. Some suggestions will be ventured, however.

(1) Let the pupils "learn to do by doing" in class under the guidance of the teacher. Happy is the teacher who may have

a supervised study period for this. Very little is accomplished by solving a verbal problem for a pupil if he himself has not studied seriously about the problem. How often is the teacher greeted by the remark, "When you explain those problems they seem just as clear, but when I get home I can't understand them at all!" Instead of solving new types of problems for the class, use the heuristic method to get the class to discover the solutions. The pupil needs individual "brain wrestling" with the problems. However, he must be taught the "holds" and "falls".

(2) Prepare by emphasizing, from the outset, the idea of "relations" between quantities (the functional relation).

(3) Stress the familiar idea of translation from English into algebraic symbolism.

(4) Point out clearly that a condition in a problem leads to a relation between the symbols or to an equation. If n unknowns are mentioned, there are n statements of fact in the problem if it can be solved. If a different symbol is used for each unknown, there will be n equations. If only one symbol is used, $n-1$ statements are needed to express the other $n-1$ unknowns and the remaining statement gives rise to the equation.

We will consider two typical problems to illustrate (4).

The sum of the three digits of a number is 20 and the digit in the tens place exceeds the digit in the units place by 5. If 594 be subtracted from the number the digits will be reversed. Find the number.

This problem involves three unknowns, the digits, and we find three independent statements of fact about them. If we use three independent (or arbitrary) symbols u, t, h in stating the problem we have one equation arising from each statement, as follows:

$$\begin{aligned} h+t+u &= 20, \\ t &= u+5, \\ 100h+10t+u-(100u+10t+h) &= 594. \end{aligned}$$

Choosing arbitrary symbols for all the unknowns requires no use of the facts given in the problem. We might as well have the same symbols for any other problem involving unknowns.

If we work with but one unknown, we need the first two statements to express the other two of our unknowns in terms of the arbitrary symbol chosen for the independent unknown. We need in this case only one equation and make use of the last of our three statements. We then have u = units digit,

$$t = u+5,$$
$$h = 20-(u+u+5),$$

and the equation:

$$100(15-2u) + 10(u+5)+u-[100u+10(u+5)+(15-2u)] = 594.$$

The angle A of a triangle exceeds the angle B by 40°; the angle C exceeds the angle A by 10°. Find the three angles of the triangle.

With three unknowns, A, B, C, we have

$$A-B = 40^\circ,$$
$$C-A = 10^\circ,$$
$$A+B+C = 180^\circ.$$

In this problem there are only two explicit statements. The other is implied in the word triangle and draws upon the student's general knowledge. Many problems are like this one in that respect. Working with one unknown symbol we need both explicit statements of the problem to express the other two unknowns, thus: $A = B+40^\circ$, $C = B+40^\circ+10^\circ$, and we use our knowledge of a triangle to obtain the equation $B+B+40^\circ+B+50^\circ = 180^\circ$.

Another vital point in the method of solution of verbal problems is a wise selection of the unknown when only one symbol is to be used. In the ordinary problem one unknown is independent of the others for its value. Pupils should learn to look for this independent unknown. It is not always the least one of the numbers to be found. In the problem above the value of t depends on finding the value of u. The digit u is independent of any statement about its particular value. In the triangle problem A depends on B and C depends on A, hence B is independent. These ideas are later harmonized with the notion of dependent and independent variables in the study of functions.

The feeling by the pupil that there is an independent symbol to be sought as a starting point, that there are exactly enough statements to translate into the necessary expressions for the other unknowns in terms of this symbol with one final statement remaining for the equation, gives him command of the situation at the outset and starts him with some confidence on his task of analyzing the problem.

(5) **Insist** that the students use a diagram if the problem

can thus be illustrated. Consider the example:

If a rectangle is 4 feet longer than it is wide and the perimeter is 32 feet, find the dimensions.

The figure for a rectangle with w (for width) marked on one end and $w+4$ marked on a side shows the problem clearly and avoids the common mistake of adding one width and one length for the perimeter, for the student sees on his figure that two widths and two lengths make the entire perimeter. In this problem w is the independent unknown, the statement about the relative lengths is the basis for the symbolic representation of the other, and the statement about the perimeter furnishes the equation, $4w+8 = 32$. If an arbitrary symbol is used for the length we have an equation arising for each of the two independent statements; namely, $l = w+4$ and $2l+2w = 32$. In a rate-time-distance problem (which is often confusing to the pupil) a diagram of the "trip" always clarifies the situation. Consider the example:

"Two men in automobiles start from points 200 miles apart at the same time and travel toward each other, meeting in 5 hours. One travels 5 miles an hour faster than the other but loses one hour on account of tire trouble. What is the rate of each?"

r r+5

A :———> time 5 hrs. ———>:<——— time 4 hrs. <———: B

:<——— 5r miles ———>:<——— 4(r+5) mi.———>:

< total distance is 200 miles————————>

The equation $5r+4(r+5) = 200$ stands out prominently on the diagram.

As a final suggestion it should be said that verbal problems given on examination should not be copies, with changed numbers perhaps, of the problems studied in class. The study of verbal problems should develop in the pupil a power of doing things with some initiative and resourcefulness, not a modicum of memorized processes for certain familiar types of problems. A real test of improvement is to give problems expressing new number relations expressed in new language. In the case of applied problems taken from everyday life they should be from a field not previously studied but so chosen that the pupils are sure to understand the conditions surrounding them.

90. Treatment of Graphs. It will be noticed that the

National Committee (Report, Chap. VII) lists graphs as one of the six topics by means of which the function concept may be taught. This ought to correct the erroneous ideas held by some that the subject of graphs is the only one in which the idea of function occurs in elementary algebra. As presented in some much used texts the subject of graphs is not even a help to clear thinking along the lines of functional relations.

Among the types of graphs that can be correctly represented on a plane[59] are bar graphs, circular graphs, broken line graphs, and graphs of algebraic functions.

The first two types mentioned do not lend themselves to an interpretation of the functional relation. For example, the number of adherents to the different religious beliefs, the populations of various countries, the relative sizes of various crops (measureable by the same unit), etc., can be expressed graphically by drawing a series of shaded lines (bars) according to some scale so that their relative lengths picture the information intended. However, the separate values represented by the lengths of the bars do not depend on any single variable. No functional relation is shown. Suppose instead that we show by a series of bars the population of a given country over a period of years. Certainly the length of each bar depends on the year to which it belongs for the population depends on the progress of time. If we draw the bars (equally spaced) perpendicular to a given horizontal line on which their bases rest and connect the upper ends of consecutive bars by straight-line segments, the broken line thus formed shows how the population increases or decreases with respect to time. Thus we have the broken line graph—an approximate expression of a functional relation.

The circular graph can only be used when some given amount is to be separated into parts. The circle represents the total amount and various sectors show the relative sizes of the parts. For example: the distribution of the annual national budget to the different departments of government; the proportionate amount of the school money of a district used for senior high school, junior high school, elementary school, and kindergarten; the proportionate amount of each dollar of tax distributed to the state, county, school district, and road fund; and many other similar problems may be effectively pictured by means of a circular graph. It does not show any functional relation.

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[59] The representation on a plane of pictures of solid bodies is often misleading.

The teaching of bar graphs and circular graphs will usually be completed in the seventh and eighth grades, leaving only the broken line graphs and graphs of algebraic functions to be considered in the course which we have traditionally called ninth grade algebra.

It is the main purpose of this section to discuss a method of presentation of graphs of continuous algebraic functions which, it seems to me, would strengthen and support the function concept rather than ignore it as many presentations in the less recent texts do. It is not in any way the intention by such a restriction of the discussion to emphasize the graphs of continuous algebraic functions more than the statistical graphs.

A conventional method of presenting the subject of graphs in algebra is much like the following: A student learns to locate points on the plane by means of abscissas and ordinates and follows this immediately with plotting of points whose abscissas and ordinates satisfy some equation, say, $3x+4y = 12$. The method of finding the pairs of values for the necessary points is usually by direct substitution of values of x in the equation as stated and consequent solution for a corresponding value of y. For example; in the equation cited above, if $x=2$, $3\cdot2+4y = 12$, and $y = 3/2$. When points enough are located the straight line through them is drawn and is called the graph of the equation. Simultaneous equations are immediately considered and the abscissa and ordinate of the point of intersection of the two lines is shown to be the solution. Such a presentation gives little or no concept of the functional relation.

A better plan is to start with the simplest type of graph of an algebraic function; namely, the graph of $y = mx$, m a constant. Practical formulas of this type are $C = \pi D$, $d = rt$, $A = ab$, $A = bh/2$. It should by all means be the first type of algebraic function studied under the topic of graphs. In plotting the graph of, say, $y = 2x$ it can be seen by studying the table of computed values of x and y that y increases twice as fast as x and easily seen on the coordinate paper as the points are located that the "steepness" of the line is 2:1 or 2. $y = 3x$, $y = \frac{1}{2}x$, and others can be used to emphasize the idea of the slope of the line or the manner in which **y varies as x changes value.** The practical formulas mentioned above should be used in the same way.

It is an easy step next to the type $y = mx + b$. Again m indicates the steepness, or slope, of the line which crosses the y—axis at (0, b). The table of values in such an equation as $y = 2x+3$ will check up with the graph in showing that y increases twice as fast as x, getting started at the value three, when x starts at zero. A notable practical example of this type is $F = (9/5)C + 32$, where F refers to the temperature on the Fahrenheit scale corresponding to the temperature C on the Centigrade scale. A carefully drawn graph of this furnishes a good chart to use in changing from one scale to the other.

The next step is to take up equations of the type $ax+by=c$, restating in the form

$$y = -\frac{a}{b}x + \frac{c}{b}.$$

Not only will the preceding method of presenting graphs of algebraic functions keep the notion of the nature of the dependence of y upon x uppermost, but it will lend itself easily to all the teaching that is usually done with respect to the graphical solution of equations. It will lay a foundation for the consideration of the function ax^2+bx+c. The use of the phrase "graph of an equation" is unfortunate and should be avoided as it clouds the issue when the graph is used in connection with the equation $ax^2+bx+c = 0$. The expression "graph of a function" is more accurate. Then a connection is easily established between those values that make the function equal to zero (called zeros of the function), the intersection of the graph of the function with the x—axis, and the roots of the equation, $ax^2+bx+c = 0$.

91. Three Attitudes Toward Teaching Graphs. One teacher (may his tribe decrease) using a typical text of the last decade but one, omits the subject of graphs, explaining to the class that it is not essential but if the class has time before the end of the term he will take it up later. He seldom finds time to return.

Another teacher faithfully and perfunctorily administers the subject in its capsule (chapter) as printed in the book as he does all the rest of the book. I often wonder if the students of such a teacher ever consider the possibility of the board of education saving that teacher's salary by assigning some one to give them an examination on the book when they have completed the study of it, say, at home.

A third teacher makes the subject of graphs a vehicle for carrying on the function concept, uses it to illustrate statistics, presents it in a way that the subject of variation is clarified, and does not confine it to one chapter alone. He so enlivens that and other subjects that the class receives an inspiration. The subject of algebra is lifted above the dry pages of the text.

Reverting to the discussion at the close of the first chapter the third type of teacher mentioned is usually (not always by any means) one especially prepared to teach mathematics who knows his subject thoroughly.

References

Ligda, The Teaching of Elementary Algebra.

CHAPTER XII

THE TEACHING OF GEOMETRY

I. SUBJECT MATTER AND OBJECTIVES.

92. General Remarks. Although geometry is taught in some form from the seventh grade of the junior high school (or even earlier) to the end of a student's study of mathematics, in most of our public schools at the present time nearly all of the course in mathematics for the tenth grade is a more or less modified form of the traditional course in plane geometry. In view of this fact, the discussion of the teaching of geometry in this chapter and those following will consider the traditional course usually given in the tenth grade. A consideration of the plan of distributing the teaching of geometry through the eighth, ninth, and tenth grades has been given in Chapter IX. What is to be said about the teaching of geometry will, in most cases, apply to the situation in any grade where the subject matter is distinctly geometry.

93. Objectives.

(1) **It should be our aim to develop powers and habits of careful, accurate, and independent thinking rather than to present geometry as a finished model of deductive logic.**

No course in the high school curriculum offers to the student so many distinct and separate opportunites to indulge in independent and original thought as does geometry, if properly presented. A theorem is to be proved. A form of proof is suggested (or given completely) in the text. If the attitude of the teacher is that the method of proving the theorem in the text is perfect and all-sufficient, the student will become a slave to authority and will probably acquire the habit of memorizing proofs. That this method has been in vogue is proved by the oft repeated expression by adults, "I never could learn mathematics for I have such a poor memory". If, on the other hand, the pupil is made to understand that there may be other proofs for the same theorem, he is encouraged to do some independent thinking. It is worth more to a pupil to discover one or more original methods, (that is, original to him) of proving the Pythagorean Theorem (scores of proofs have been published) than to learn in a letter-perfect manner the proof used by Euclid, or to have pointed out to him the sequence of logical steps in the old familiar series of lemmas (called theorems) leading up to theorem, "The area of a rectangle is equal to the product of the base and altitude." He is willing to accept this theorem as readily as the postulates.

So much for the theorems usually proved. In the original exercises, whether they are merely minor theorems relating to formal geometry or applications to ordinary human experience, the pupil **must** think for himself or he can not succeed. The whole course, with the exception of the definitions, postulates, and axioms, and the facts stated in the theorems, is one in which the pupil should be encouraged to think rather than memorize; in which he should be taught to rely on himself; in which his initiative should be continually called into action; in which he should acquire, by continual repetition of the act, the **habit** of examining given facts and conditions with a view to forming his own conclusions rather than appropriating the conclusions of others. The cultivation of the habits of self-reliance and independent thinking in a youth certainly outranks in value the presentation to him of a perfectly coordinated body of truths in logical sequence for the sake of the scientific facts involved or for the show of deductive logic.

It is obvious that it is not at all necessary to depart from a logical sequence of theorems nor to violate the spirit of deductive logic in geometry in order to accomplish the results just mentioned.

(2) **It should be our aim to connect the geometric theory, by means of practical exercises, numerical and constructive, with all the applications of the subject we can discover.**

It is the experience gained in doing original exercises of this type that makes the pupil's knowledge of the subject useful to him. Many a pupil in the past, with an understanding of the meaning and necessity of postulates and axioms, has been able to give with logical understanding the complete proof of a theorem, step by step, and not realize that the forms manipulated by him may have numerical values attached and can be applied to simple practical problems.

For example, the theorem, "The ratio of the areas of two similar polygons is equal to the ratio of the squares of any two corresponding sides", has been proved by pupils who could not at the conclusion of the proof answer the question, "If a polygon with sides 4 inches, 7 inches, 9 inches, 6 inches and 8 inches long, has an area of 40 square inches, what are the lengths of the sides of a similar polygon with an area of 10 square inches?"

Again, the theorem, "If a tangent and a secant are drawn to a circle from a common point outside the circle, the tangent is a mean proportional between the entire secant and the external segment", has been proved by pupils who apparently understood the logic of the proof, step by step, but who could not at the conclusion of the proof answer the question, "If in the adjoining figure AB = 4 inches and BC = 12 inches, how long is AT?"

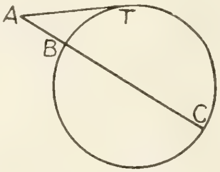

FIGURE 16

A pupil can know perfectly the definition, "Two angles whose sum is equal to a straight angle are called supplementary angles; either angle is the supplement of the other", and not see that the relation is also expressed by a° and (180—a)°. How many of the boys who prove so easily the exercise-theorem, "The diagonals of a rectangle are equal", realize that if they want to lay off a rectangular tennis court, they may, with the aid of a steel tape, check the construction by measuring the diagonals?

It is not meant that in the applications of geometry to problems of trade and industry we shall attempt to show the students how to solve every kind of problem that may arise in their later experiences, but that enough of it shall be done that they may realize the usefulness of this tool called geometry. In transferring from the formal statements of the theorems to the practical applications and in making the general knowledge fit a special case there is opportunity for the pupil to exercise his ingenuity and do some independent thinking. Thus in the pursuit of the second objective we aid in reaching the first.

(3) **We should try to make our pupils appreciate the worth of the subject in its vital connection with the development of useful learning in the progress of civilization.**

By the introduction of historical matter we should make our pupils see the worth while contribution that geometry has made to scientific progress from the time the ancient "rope stretchers" of King Sesostris laid out the right angles of their land surveys along the banks of the turbulent Nile to the present day when Einstein, in forming a theory to explain the mysteries of the universe, assumes that figures may **not** be moved about in space without change of shape or size. They should see its contributing influence on art and culture to such an extent that they may feel that it is as much worth while to know about it as to know about many other subjects in the curriculum that do not contribute to their ability to earn a greater wage next year.

In conclusion, let this be said. In reaching the first objective we conserve all the values geometry ever had as a mind trainer. Keeping the other two aims in mind and conscientiously striving to reach them will forestall the question, "Why should I study geometry? Means of attaining these objectives will be discussed in later chapters on methods.

94. The Content of the Course in Plane Geometry. In a later section (§98) the change in texts is discussed as an evidence of the change in content of the course. Since text book writers do not agree, the question at once arises, "What shall be included?" As mentioned in the beginning of this chapter, we consider the traditional tenth grade course in plane geometry in order to serve those who, because of existing conditions, are teaching such a course.

The National Committee (Report, Chap. VI) has recommended for plane geometry a minimum list of 52 fundamental theorems and 19 constructions[60]. A subsidiary list includes 34 theorems, making a total of 105 "propositions". The College Entrance Examination Board in Document No. 108 (May 15, 1923) lists in its requirements 89 theorems and 20 constructions. Thirty-one of the 89 theorems are selected to form a special list from which theorems to be proved in the examinations will be chosen. Comparison of either of these lists with any modern text written before the report of the National Committe will show the trend of sentiment toward a greater emphasis on original exercises and a reduction of time spent on model theorems.

No individual can assume the authority to say just how many or what theorems shall be included, or how much of the time should be spent on original exercises. Much value can be placed, however, on the suggestions of a group like the National

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[60] See Appendix, II.

Committee, or the Commission on College Entrance Requirements in Mathematics, whose report to the College Entrance Examination Board was the basis for the selection by the Board of its recommended list of basal theorems. Each of these two groups contains representative teachers from both high school and college ranks, and from various sections of the country.

Certainly the content of the course should be determined by the answer to the question, "What will help the most in reaching the desired objectives?" If a study by the pupil of more than 130 completely proved model propositions is a better way to develop in him initiative, originality, and the habits of self reliance and independent thinking than the proof by him of a greater number of original theorems and the solution of numerous problems that require an insight into the subject deep enough to relate the general facts to special cases, then certainly the emphasis should be put on the study of the model theorems. Otherwise it should not. If a study of original exercises which are but minor theorems of formal geometry will improve a pupil's ability to apply his geometry to problems of human experience more than a study of numerous practical problems typical of the problems of trade and industry, then let his attention be confined to the type of original exercise popular fifty years ago. I do not believe that any teacher needs help in deciding between the alternatives presented in the last two statements.

With the National Committee's and the College Entrance Examination Board's lists to guide him, a teacher who must by force of circumstance use a textbook containing 130 proved propositions can eliminate one-third of them, using the time thus saved in training his students by trial on original proofs to think independently, use initiative and ingenuity, develop self reliance, and make practical application of the facts learned. Time may be had to stress the computational side of mathematics, in which our pupils are never strong enough, and to introduce in connection with similar triangles the use of the trigonometric ratios.

II. HISTORICAL DEVELOPMENT OF METHODS.

95. **The Pre-Euclidean Epoch.** There is nothing on record concerning the teaching of geometry in ancient Egypt and Babylonia. There was little at that time that could be called geometry and it was composed of practical methods like the rules of mensuration. These rules may have been passed from one to another like an artisan today learns new "rule-of-thumb" methods

from his companions. In Babylonia the knowledge was passed from one astronomer to another, in Egypt from one priest to another.

The teaching of geometry in Greece was linked with the teaching of philosophy. Plato made it a prerequisite to the study of philosophy in his school. The Sophists taught it in Athens, using Socrates' method of question and answer. It is pertinent to remark that the first authentic method of teaching geometry was NOT a method by which the learners were asked to study model proofs in order to learn logic, but a heuristic method by which the students were led to develop the proofs of the theorems. Much of the teaching was done in the open, the figures being drawn in the dust. It is probable that great teachers like Plato and Pythagoras lectured to a select group (all mature). The members of the "inner circle" in the school of Pythagoras at Crotona, in Southern Italy, were sworn not to divulge the precious and secret knowledge given out by the great master.

The idea of motion in geometry (like the definition of an angle by rotation) was entirely foreign to Greek geometry. The science of algebra not having been developed, their methods were exclusively geometric and for that reason sometimes laborious. Not even in the theory of proportion could they break away from the geometric concepts and methods.

The early Greeks are credited with giving us methods of analysis and the "reductio ad absurdum".

96. From the Time of Euclid to the Discovery of the Art of Printing. The arrangement of the theorems of geometry in a logical sequence by Euclid was a distinct contribution to the teaching of geometry though his labors did not bear abundant fruit for a thousand years after his death. The most noticeable immediate effect of the "Elements" seems to be that it created a feeling that the development of pure geometry might rest, for the emphasis was placed on the practical side; that is, on the applications of geometry. Archimedes (see §45), the first great geometer after Euclid, gave attention to the application of geometry to mechanics. He frequently gave mechanical proofs for some of his propositions and may be regarded as being the first teacher to use a laboratory method. What little attention the Romans gave to geometry was mainly on the practical side.

It has been mentioned elsewhere that it was the Arabs who preserved the "Elements" and gave it back to Europe. The translations of the "Elements" from Arabic into Latin in the twelfth century gave a great impetus to the study of geometry in the church schools. An apathetic attitude toward a neglected subject which had degenerated into a few simple constructions and rules of mensuration was changed to one of keen interest. Inasmuch as the art of printing was unknown, the only method of teaching was by the lecture method.

In the thirteenth and fourteenth centuries the universities began to require a certain knowledge of "Euclid" for a degree.

97. From the Discovery of the Art of Printing to the Present. Toward the end of the fifteenth century the first printed edition of Euclid was published and from then on the study became more popular. A popular reaction in the sixteenth century against the study of purely theoretical geometry for the sake of mental discipline threatened to turn the whole subject back to practical rules of mensuration and similar applications but the next century saw the "Elements" predominant as the foundation for the course in geoemtry.

Not until the eighteenth century was geometry very generally taught in the secondary schools of Europe, and in England not until the nineteenth century. Geometry was retained even longer as a university subject in the United States. Harvard did not require plane geometry for entrance until 1844. Plane geometry has been considered distinctly as a high school subject in the United States not much longer than half a century. Twenty-five years ago it was generally placed in the high school curriculum as an eleventh grade subject though in 1892 the "Committee of Ten" recommended a high school course that placed plane geometry in the tenth grade. For the last fifteen or twenty years it has been taught mainly in the tenth grade when taught as a separate and distinct subject. The advent of the junior high school has had a tendency to bring the teaching of some demonstrative geometry down to the ninth grade and has augmented the inclusion of intuitive and constructive geometry in the seventh and eighth grade courses in mathematics.

98. The Evolution of Textbooks in America. The content of the course is usually determined by the text and the method of teaching is more or less governed thereby. It is therefore pertinent, and very interesting, to trace the change in the nature of the geometry text since the introduction of the subject into the American secondary school.

The high school text in "Euclid" of fifty or sixty years ago differed very little in content from the "Elements" that had been studied by university students from the time of the Renaissance. A certain typical text (copyright, 1862) contained (besides the definitions, postulates, and axioms of geometry) the successive "books" of propositions, in logical order, with proofs carefully and completely given. There were 181 proved propositions consisting of 139 theorems and 42 problems. On the final pages of the book there was assembled a list of "Exercises in Euclid", 417 in all, which were other theorems of pure geometry.

Twenty-five years ago the typical text had reduced the number and changed the arrangement of the propositions, having five "books" for plane geometry, each being concluded with a set of original exercises provable by means of the theorems of the "book" or preceding "books". The exercises were, as in the earlier type of text, merely minor theorems of formal geometry. A very much used text (copyright, 1899) introduced the subject with fourteen pages of definitions and axioms, defining such terms as "scholium" and "perigon". No illustrative matter, except figures, was considered in connection with the definitions, the bare definition being given with logical exactness. The student was told that "A straight line is such that any part of it, however placed on any other part, will lie wholly in that part if its extremities lie in that part." The definitions, postulates, and axioms were followed immediately by the first theorem, "All straight angles are equal". There were 167 propositions (including 126 theorems and 41 problems of construction) and 603 exercises. The intention of the author (and the teacher) was to train the pupil in logical thinking by a study of the 167 perfect and completely proved examples of logical demonstration so that he could "go and do likewise" when the exercises were considered.

A book published in 1912 that is used in a great many schools at the present time contains 133 proved propositions (including 100 theorems and 33 problems), 400 exercises that relate to abstract or formal geometry and 470 other exercises that include simple questions, solutions of simple equations, and about a score of applications to trades and industries.

More recent texts reduce the number of exercises dealing with abstract geometric forms and increase the number of practical applications. Some sacrifice logic to geometric insight when the general geometric interests of the student would thereby (in the opinion of the authors) be better served. Some take away the emphasis from the logical proofs of a sequence of standard theorems and place it on experimental proofs (?) and practical applications to problems of trade and industry. In all of them the introductory list of definitions is materially reduced, all definitions possible being postponed until actually needed in theorems. Many definitions are accompanied by illustrative examples. The earliest propositions of the text of a half century ago, which for the sake of logic were so carefully demonstrated (but whose truths were so obvious to the pupil that the demonstrations were confusing), are now listed as preliminary propositions, many of which are to be accepted in the same spirit as the postulates and axioms immediately preceding them. Following every proposition, or group of related propositions, is a list of exercises and thought-provoking questions, those at the beginning requiring simple one-step proofs. Many of the easier theorems in the sequence of propositions are left unproved with possibly a hint or two as to the method of proof. Many others leave several steps to be completed. More algebraic method has been introduced. Many unnecessary technical terms have been omitted. Some teachers of geometry do not know what a perigon is and few know what is meant by the term, scholium.

The latest texts, following the recommendations of the National Committee, materially reduce the number of theorems, include more practical exercises, give considerable attention to the trigonometric functions, use more algebraic methods, and replace by suggestions more and more of the statements in the printed proofs.

99. Changing Methods of Teaching Geometry. The method of asking pupils to learn a synthetic proof printed in a book and then testing by a recitation method whether or not they have learned it well is dying. It was born in the days when the students of geometry had mature minds and could read and appreciate logical proof. The method for a long time in Germany and France has been to encourage the pupils in the development of the proofs of new theorems. Good teachers in the United States have long used the same methods in spite of the handicap of the texts. Some have boldly thrown the text overboard and used a syllabus. The texts are now undergoing a change which it is hoped will encourage more and more the use of the analytic and inductive methods. Indeed we may eventually return to the primitive methods of the Greeks before Euclid when scientific geometry was being born, and permit our pupils the privilege so long denied them; namely, the experience of discovery of the theorems anew, the living over again of the experience of the race in its development. When that era comes we will no longer need to defend geometry as a disciplinary study as has been done so much during the era of "proof learning"—results will defend it.

References

Stamper, A History of the Teaching of Elementary Geometry.

Smith, The Teaching of Geometry, Chaps. II-VII.

Schultze, The Teaching of Mathematics, Chap. II.

CHAPTER XIII

THE INTRODUCTION TO GEOMETRY

100. The Definitions. I remember vividly my first lesson in geometry. It consisted of several pages of formal definitions, axioms, and postulates which were to be memorized. The definitions of practically all the terms used in the first fifty pages were classified in the first dozen pages. Even the postulate of parallels was learned in the first lesson, though it was not used for some weeks afterward. A much better arrangement is found in the average modern text. Exercises and questions designed to provoke independent thought about the new terms whose definitions are being studied are interspersed with the definitions. The parallel postulate is not encountered until the time comes to take up the study of parallel lines. There are few texts, however, that do not introduce more definitions than are necessary in the beginning. For instance, why should theorem, corollary, and other such terms be introduced before we are ready to use them?

If a teacher must by force of circumstances beyond his control use a text in which the introductory pages are clogged with definitions of terms not used for many pages later in the book, he should select in the beginning only part of the definitions in the book, those whose immediate use not only justifies their introduction but emphasizes their meaning. The remaining definitions should be studied later, just when the terms defined are about to be used.

Every definition should be given a numerical illustration if possible. The statement, "two angles whose sum is a straight angle are called supplementary angles", can be memorized and understood as it stands, but it is more readily understood by the pupil and the meaning is made more vital if he is led to consider pairs of angles like 140° and 40°, 120° and 60°, etc., not in a list of exercises on a later page, but in connection with the definition.

After each group of related definitions there should be exercises which give the student opportunity to become acquainted with the terms just learned by using them. For example, the notion of angle is made clear and the definitions of various kinds of angles are given. Immediately following this should be a group of exercises related particularly to the facts about angles just learned. They should be such as will not only emphasize the meaning of the terms but will call forth some independent thought about them. The following partial list illustrates the nature of such exercises.

1. What is the complement of 33°? 59° 30′? a°? 2x°?
2. What angle is equal to its complement?
3. An angle is four times its complement. Choose a symbol for the angle, form an equation, and solve it. What is the angle?
4. What angle diminished by 16° is twice its complement?
5. The difference between two complementary angles is 28°. Find the angles.
6. What is the supplement of 25°? 78° 30′? b°? 3y°?
7. The ratio of two supplementary angles is 7:11. Find the angles.
8. The supplement of a certain angle is three times its complement. Find the angle, the complement, and the supplement.
9. Can an obtuse angle have a complement?
10. Is the supplement of an angle always greater than its complement? How would you prove it for any angle, say, a°?
11. Is the difference between the complement and the supplement of any angle always the same?
12. Did you consider negative angles as a possibility in your answer to Ex. 9? Would you change your answer if negative angles are to be considered?
13. What is the supplement of a right angle? The complement?

This list of exercises, with definitions of angles preceding it, should not be merged with the general introductory matter. It should be a distinct unit. In a similar manner should all terms necessary for immediate use be introduced. If the text does not provide this, the teacher should supplement the text.

The logical situation with respect to undefined terms was considered in the first chapter (see §4). It is not necessary to have the pupil comprehend this situation. Such terms as point, straight line, etc., may be discussed (not defined) in the beginning as the fundamental elements of geometry. Their nature should be clearly illustrated.

101. The Introduction to Logical Proofs. I also remember my second lesson in geometry (or was it the third?). After mastering definitions enough to carry us well into the first "book", including the definitions of "scholium", "contradictory theorem", "converse", etc., we proceeded to prove that "All straight angles are equal". One might say that having swallowed the camel of the parallel postulate we strained at the gnat of equal straight angles. Any healthy minded youth who is encouraged to think about it will be more inclined to doubt the statement that only one line can be drawn parallel to a given line through a given point outside the line than he will the statement that all straight angles are equal.

The average modern text aims to avoid this predicament of having the student impede his progress by quibbling over minor and ultra-logical technicalities. The theorems whose truths seem axiomatic are placed immediately after the axioms with little or no proof. Sometimes they are boldly listed as postulates. Why not? What shall limit the assumptions we may make at the beginning of our logical sicence? If our desire is to make it a perfect type of a logical science then our set of postulates must satisfy three conditions. They must all be independent; that is, no one may be proved by means of the others; they must be non-contradictory; they must be such that they exactly define this science and no other. To attempt to make them fulfill the first condition is to limit them to the irreducible minimum and undo all efforts to make geometry comprehensible to the tenth grade youth. This is the condition we have been fighting for years. The skilled logician may do his work with but few tools, but the untrained youth needs more, hence we lay down more assumptions at the beginning. It is easy to see that the other two conditions are satisfied, even if we assume at the outset the truth of many of the preliminary theorems. Within the limits of comprehension of our students we are proceeding in the spirit of mathematical and logical development by adding these theorems to our list of postulated facts. One of Euclid's postulates was that "All right angles are equal". In some cases the preliminary theorems are called corollaries and the simple steps are suggested by which their proofs follow from the axioms and postulates just preceding.

Having avoided the possibility of dulling the pupil's interest by asking him to engage in tedious logical quibbling over evident truths, the usual custom is to initiate him into the mysteries of a logical demonstration by means of the superposition proofs of the congruence theorems. What an initiation! The technical details of a formal proof of the theorem, "If two triangles have two sides and the included angle of one equal to two sides and the included angle of the other, respectively, the triangles are congruent", are not the things that will give a student a proper appreciation of geometry at the outset. The necessity for some of the minor logical steps in the proof are as far removed from his comprehension as is the necessity for proving "all straight angles are equal". The proof is too long for a beginner. Having prevented the vessel from foundering on the Scylla of logical finesse, we should not let it be engulfed in the Charybdis of complexity. Let the first theorem be one with a proof involving a few simple steps each of which demands a reason for being in the proof. This congruence theorem and its dual, "If two triangles have two angles and the included side of one equal to two angles and the included side of the other, respectively, the triangles are congruent", might well be accepted (postulated) in the beginning as facts (to be proved later, after the student has had experience in formal demonstration). A scheme like the following might be used:

After defining and classifying triangles, do the following exercises:

1. Draw a triangle with an angle of 55° (use protractor) making the two sides including the angle one inch and one and one-half inches, respectively.

2. Draw a triangle with a side one inch long, making the two angles adjacent to this side 30° and 57°, respectively.

3. Draw two triangles as follows: One has a two-inch and a three-inch side including an angle of 60° and the other a two-inch and three-inch side including an angle of 80°. Cut these triangles out and try to fit one upon the other to test their congruence.

4. Make two triangles with the following specifications and try to make them coincide: sides two and one-half and three inches with included angle 60°; sides two inches and three inches with same included angle.

5. Perform a similar experiment with two triangles, each having sides 2 inches and 3 inches long and the included angle 60° in both cases.

Which of the pairs of triangles compared in Exercises 3, 4, and 5 seem to be congruent?

6. Try fitting together two "cut out" triangles one of which you have constructed with two angles 60° and 45° having a side 2 inches long included between them and the other with angles 60° and 45° including a side one and one-half inches long.

7. Perform a similar experiment with two triangles made according to the following specifications. Angles 60° and 45° including a side 2 inches long; angles 80° and 45° including a side 2 inches long.

8. Make a similar experiment with two triangles, each having angles 80° and 45° including a side two and one-half inches long.

Which of the pairs of triangles considered in Exercises 6, 7, and 8 seem to be congruent?

Assumptions about triangles. The experiments performed in Exercises 3-8 above lead us to suspect that:

A. If two triangles have two sides and the included angle of one equal, respectively, to two sides and the included angle of the other, the triangles are congruent.

B. If two triangles have two angles and the included side of one equal, respectively, to two angles and the included side of the other, the triangles are congruent.

We need the knowledge of these facts in this chapter to better acquaint us with the methods of geometry. We shall therefore assume for the present that they are true. *They will be proved later.*

After these facts are accepted there is possible for the pupil a number of simple one-step exercises applying them. Consult any good modern text for illustration as to the nature of these exercises. If the exercises are well chosen and carefully graded, the student may be led gradually up through the experience of simple deductions to an appreciation of the meaning of a proof by actually making proofs, though they may not be formally labeled as such. He can see the wonderful applications of geometry through exercises like the method of Thales in measuring the distance of a ship from the shore (§41). All of this can be done before a pupil has had a formal introduction to the axioms and postulates. He will then have a clearer comprehension of the necessity for and meaning of the postulates when they are presented. A teacher can easily supply this kind of an introduction to geometry if it is not provided in the text.

When we are ready to introduce our students to a formal demonstration, we should have a theorem with two particular attributes. The fact stated should seem to require proof, and the proof should be one that is easily followed, each step appealing to the pupil as being a natural and necessary thing to do. Two examples are offered here:

(1) "The bisectors of two supplementary adjacent angles are perpendicular to each other."

(2) "The angles opposite the equal sides of an isosceles triangle are equal".

102. Intuitive and Constructive Geometry. By intuitive geometry is meant the study of geometric forms for the purpose of becoming acquainted with their nature by using them and experimenting with them, and consequently making conclusions as to relations existing between them which seem to be true. In such an introduction to geometry considerable use is made of the ruler and compasses, the tools of precise constructive geometry, and the protractor. In many of the modern courses in the junior high school, intuitive geometry is introduced as early as the seventh grade and continued through the eighth and ninth. If students have not had intuitive geometry preceding the usual tenth grade course in plane geometry, it is necessary to emphasize it in the introduction to the course. Three good methods of presenting it are: (1) by exercises in geometric construction, (2) by lists of thought-provoking questions and exercises, making use of the mathematical equipment (arithmetic and algebra) of the pupil to familiarize him with the forms of geometry, and (3) paper folding.

In the first reaction against the traditional method of presenting plane geometry as a strictly logical subject from the beginning, as it had been taught to college students for centuries, the "construction" method of approach was featured very much by some. Pupils used the ruler and compasses to make precise construction of perpendiculars, parallels, line-segment-bisectors, angle-bisectors, triangles with three parts given, inscribed and circumscribed circles, etc. The probable result of such an approach to geometry, unless guided by a wise teacher, is that the pupil will assume geometry to be much like a course in mechanical drawing. Disillusionment is sure to come to him, but it is likely to come after he has "lost his step" in the course. There is a valid objection to requiring a pupil to do a great many complicated exact mathematical constructions before being able to prove them correct.

Examples of the second method of informal or "intuitive" introduction to geometry may be found in many of the most recent texts and in the list of exercises relating to angles in §100. The following exercises are given for added illustration.

1. If a basin of water is allowed to become absolutely still, what form will be taken by the surface?
2. How would you find the shortest distance from Chicago to New York?
3. How many straight lines may be drawn through three points? Through four points? Through five points?
4. How does a man proceed if he wants to set three or more posts in a straight line?

The method of paper folding has been discussed in the section on the laboratory method.

103. Notation. A high school principal once visited the room of a geometry teacher during the first month of the course and in questioning one of the pupils at the board found her a bit uncertain about the method of reading an angle by use of three capital letters, the middle one being at the vertex. Later, the principal criticized the teacher for failing to teach the "fundamentals", referring to the incident just related as an example. That teacher in the beginning of the course in plane geometry used consistently the method of designating an angle by one lower-case letter placed between the sides near the vertex (or by a number so placed) using the traditional Euclidean three-letter method only when necessary to avoid ambiguity, expecting the pupils to acquire familiarity with it by gradual use. Was the principal, or the teacher, right? Is the ability to read angles by any particular method one of the fundamentals? Since only ONE CONCEPT is involved in the term "angle", why should a cumbersome method of THREE SYMBOLS be "fundamentally" needed to represent it? There are cases where, to avoid ambiguity, it must be used; but until a student begins to use complicated figures the work can be materially simplified by use of the simpler one-symbol method. Resorting to the one-symbol method at all times in the course where no ambiguity arises will save time.

For the same reason the use of a lower-case letter to represent a line segment is desirable. For example, the altitude of a triangle may be more simply represented by h instead of the two capital letters at the ends.

In treating pairs of figures that have corresponding parts, like congruent or similar triangles, the use of the same letters for corresponding points with primes or subscripts on one set is desirable. For example, designate two similar triangles as ABC and A′ B′ C′. Some textbook writers in the past have perpetrated the monstrosity of two such triangles lettered ABC and DEF, presumably with the intention of making the pupil think and avoid the memorization of proofs. There are too many good and simple ways of avoiding memorization for one so deliberately to confuse a pupil beginning a subject so new and strange as geometry is to the tenth grade pupil.

References

Schultze, The Teaching of Mathematics, Chap. VI.
Smith, The Teaching of Geometry, Chap. IX.
Young, The Teaching of Mathematics, Chap. XIII.

CHAPTER XIV

THE ORIGINAL EXERCISE

104. Introductory Remarks. The real test of a pupil's understanding of or his acquired ability in geometry lies in the solution of the original exercises. This is one of the most difficult parts of the course for the inexperienced teacher to teach.

The parents of a fine baby boy, their first one, were told by the doctor that when the baby was seven months old they might begin to give him other food than milk. Among the foods mentioned for a baby under one year of age was coddled egg. This, being easily prepared, appealed to the parents and on the very day the baby was seven months old he was given his first feeding, a whole egg! The egg was eaten but not assimilated. Before the organs of assimilation could get hold of the egg it returned.

Many pupils in the past have been put through one kind of "diet" (learned proofs) in geometry for seven months and then expected to bolt some original-exercise "eggs". They could memorize them but they could not assimilate them and they were in most cases soon gone from the memory. In the more recent past pupils were supposed to train themselves on model proofs for a whole chapter and then "go and do likewise" in the originals at the close of the chapter. Suppose a football coach lectured to his team, having them study diagrams and memorize model plays until they were letter perfect in the theory, and then, without any scrimmage, sent them into a game! As the coach develops, first by easy stages and then by successively harder stages of practice, the muscular equipment for the contest, having each new play practiced by the team as it is taught, so may we develop "mental muscularity" in our geometry pupils. It is an application of the old saw: "Learn to do by doing". The exercises should be approached by easy steps and **must be graded,** beginning with one-step exercises. This is done in most texts now and the teacher is relieved of that burden. However, there is left for the teacher the task of preparing the pupils for this part of the work by the use of thought-provoking questions as the class proceeds with the early theorems. These theorems proved in the text can be developed in a manner conducive to original and independent thinking rather than merely assigned in advance for the pupils to learn (memorize).

105. **An Excessive Number of Model Proofs.** A tradition of Medieval Euclidean geometry which clings to us yet is that a student should have a great number of model proofs and that skeleton proofs at least should be suggested for all theorems emphazied by being called propositions. We should not be amazed then if teachers complain that it is impossible to break the habit of memorizing. A certain child persisted in the habit of using his dessert spoon instead of his fork through all the dinner courses as the only means of conveying food to his mouth. His parents stopped the habit by the simple expedient of removing the spoon until time for dessert.

Model proofs were written by Euclid into the text from the beginning for mature scholars to study just as "translations" of Greek and Latin are made for scholars to use. The use of the model proof will be abused by the student just as he abuses the use of "translations" in a foreign language. The situation in mathematics is even worse for he does not need to do it surreptitiously as he is for the most part encouraged to do it by the teacher's attitude toward the printed proofs. The best way to stop this habit is to "remove the spoon (proof)".

When a student has studied one model proof of a congruence theorem (say, "two sides and the included angle of one triangle equal to two sides and the included angle of the other, etc.") why should he then be presented with the completely proved model of another nearly like it? Would it not encourage initiative and independence more to allow him to model the second proof himself, with the help of enough suggestions and questions to guide him to the proper conclusion? When a model proof of the "reductio ad absurdum" type has been once presented, why should not the pupil be expected to follow this model in shaping all further proofs of that type, rather than having the same general argument presented with a different figure and different names each time? Suggestions should of course be given. It is the consideration of questions like these that has caused teachers in the past to refuse to use the customary text, substituting for it a syllabus of the course containing the definitions, postulates, and statements of the theorems, etc., to be considered. The pupils are encouraged to develop their own proofs for the theorems, aided by the kindly suggestions of the teacher. They learn how to prove theorems by proving them rather than by studying models.

Many teachers will hesitate before pursuing such a bold course, preferring to use a text. How then may we get the best results in the way of encouraging independence and originality when all the propositions are proved in the text?

106. Getting the Maximum Value out of Proved Theorems. **Develop the propositions with the class as exercises in independent thinking before assigning them, using the heuristic method.**

After a class has been guided through a new theorem by a teacher who can use the heuristic method cleverly, the pupils should be encouraged, and should have confidence enough, to attempt to follow the method of laying a paper over the textbook proof, covering everything from the line marked "Proof" on to the end, and with the figure and statement of the theorem before them make all, or at least a great part of, the demonstration before referring to the textbook proof. This method of procedure will minimize the curse of memorization if the pupils can be persuaded to follow it. The attitude of the teacher toward the course can be such that the pupils will feel that they must use the method or they are not properly fulfilling the requirements of the teacher.

Any method of procedure that calls for a considerable amount of thinking in the study of the proved propositions will pave the way for a successful attack on the original exercises. Indeed, a treatment of the propositions that demands initiative and originality fulfils as far as it goes the purpose of the original exercises and minimizes the need for certain types of original exercises; namely, the "minor-theorem" type. What is said in the following sections with regard to the method of learning original exercises applies also to a heuristic study of the propositions.

107. Using the Hypothesis. One very vital reason for the many failures in the proofs of original exercises is **the failure to use the complete hypothesis as a working tool.** This will be illustrated if the reader will stop and prove this exercise: If perpendiculars AM and BN are drawn from the vertices A and B of a scalene triangle ABC to the opposite sides BC and AC, respectively, the feet of the perpendiculars (points M and N) are equidistant from the middle point P, of the side AB. After you have proved the exercise just stated try this next one until you have either proved it or are willing to give up. The ratio of the minor to the major arc of a certain chord of a circle, center O, is 1:2. On the diameter parallel to the given chord are located two points, P and Q, so that PO = OQ = one fourth the diameter.

Perpendiculars are erected to the diameter at P and Q, intersecting the chord at M and N, respectively. Prove ∠MON is a right angle.

The use of the first exercise time after time in classes composed of teachers of mathematics and university students preparing to teach has brought out the fact that much time is frequently wasted trying to prove the congruence of the triangles APN and BPM instead of attacking the right triangles at once, thus making use of the expressed condition of perpendicularity. In other words, the explicit statement in the hypothesis is frequently ignored. Any attempt to prove the exercise without including the fact of right angles at M and N is futile, yet the majority of high school students (and some teachers) might fail to use it for a long time. When one sees the two right triangles on the common hypotenuse, AB, he realizes at once that the vertices of their right angles are equidistant from the middle point of their common hypotenuse.

The second exercise was set in an interscholastic contest in geometry where the contestants had been chosen by competitive methods and were supposed to represent the choice mathematical students of the high schools of a state. Ninety per cent of the contestants who tried it failed to make use of the fact that the chord cuts off one third of the circle and is therefore a side of an inscribed equilateral triangle which bisects the radius perpendicular to it, making PM = QN = PO = OQ, from which follows the proof. Some of the contestants submitted "proofs", getting PM = PO and OQ = QN without even taking into consideration the arc-ratio, 1:2, which was equivalent to establishing the fact of a right angle at O for any, and therefore every, chord that may be drawn.

These two exercises just given illustrate the following point.

No fact deserves more emphasis in teaching how to attack new theorems than this: Fix the hypothesis in your mind and use it.

If a theorem can be proved without any given statement in the hypothesis then that statement should be omitted. To ignore the hypothesis, or part of it, is like trying to walk across the river on the water while the ferry boat pulls out beside you; like carrying your pick and spade to the rocky trench and laying them down beside you as you make futile efforts to dig with your hands.

108. Aiming at the Conclusion. The next most important thing to emphasize is that **the conclusion must be kept in mind.** If a touchdown is to be made the team must keep in mind the direction of the goal and continually strive to advance the ball in that direction. The goal can not be reached unless the five yard line can be crossed. If the team can get "first down" on the five yard line, it may reach the goal. Similarly, the ten yard line must be reached before the five yard line. So, in attacking an original theorem, fix the conclusion in mind and find some fact about the figure from which, provided it can be established as true, the conclusion can be reached by one step. Next, find what is necessary to establish the truth of this fact, and so on. In other words, work from the conclusion back toward the beginning. This was discussed under "Analytic Method" in Chapter VI. Sometimes it is best to work from both the hypothesis and conclusion until the lines of reasoning meet on common ground. Teach the class to "keep one eye on the hypothesis and the other on the conclusion".

109. Drawing the Figures to Scale. The weakest indictment that can be brought against a teacher who allows the pupils to struggle along with carelessly or incorrectly drawn figures is that he is guilty of gross neglect. A carelessly drawn figure fails to render proper assistance in the proof. The pupil loses help that he might otherwise have. An incorrectly drawn figure is positively misleading and a student by so drawing his figure places a handicap on himself.

As an illustration, suppose we want to prove that the altitudes on the equal sides of an isosceles triangle are equal. Suppose we carelessly draw a triangle that does not have two equal sides, as in Fig. 17, where, according to the hypothesis, AC is supposed to be equal to BC. The two triangles, ABM and BAN, that must be proved congruent do not appear so in the figure and one must prove what seems untrue. If, instead, we draw with some care an isosceles triangle as in Fig. 18 the figure

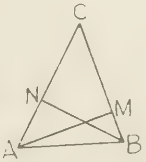

FIGURE 17

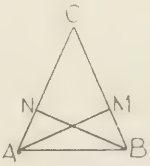

FIGURE 18

suggests by its appearance that triangles ABM and BAN are congruent and we immediately begin work on that clue, using the conditions in the hypothesis in an attempt to establish the relation suggested by the figure.

A more prevalent error is the one opposite in nature to the one just mentioned; namely, drawing a special kind of triangle or other figure when no special conditions have been given in the hypothesis. Both forms of error in drawing figures must be avoided.

In drawing figures for theorems, exercises, etc., the protractor is better than the compass for drawing angles, perpendiculars, angle bisectors, etc.

110. What Kind of Original Exercises Should Be Used?

(1) Minor theorems of formal geometry.

Many of the valuable and interesting theorems of formal geometry that can not be given the prominence of propositions will naturally be included among the exercises. Examples of these are:

(a) The lines joining the middle points of any quadrilateral form a parallelogram.

(b) The diagonals of a rhombus are perpendicular to each other.

(c) The middle point of the hypotenuse of a right triangle is equally distant from the vertices of the triangle.

(d) The bisectors of the angles of a triangle meet in a common point.

(e) The orthocenter, the centroid, and the circumcenter of a triangle are in the same straight line.

In the texts of more than fifty years ago we find few, if any, exercises other than theorems of formal geometry. As the list of propositions is reduced, more of these theorems will be added to the list of originals. The proofs of the more difficult of this type of originals always have been and always will be a challenge to the initiative, originality, ingenuity, and reasoning ability of the best members of a class. They are "meat" for the brilliant but "poison" for the low grade students. However, many of the less difficult ones are suitable for all grades of students and express geometrical facts that are very helpful in extending the field of application of the subject.

(2) Simple exercises in formal geometry.

There are many simple truths, which are really theorems requiring but one or two steps in the proofs, that can be stated about the purely formal relations of arbitrary geometric figures. They can be graded so that a pupil may gradually acquire by experience the ability to make proofs. A pupil can learn more easily how to demonstrate theorems through experience with graded exercises than by the study of model proofs of complex theorems. The following exercises illustrate the type.

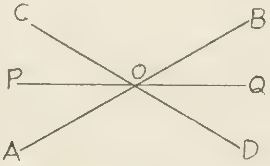

FIGURE 19

(a) In Fig. 19 PQ bisects the angle BOD. Prove ∠COQ = ∠AOQ.

(b) In the same figure prove that OP bisects the angle AOC.

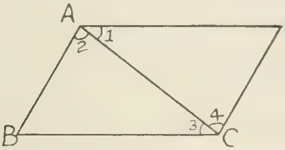

FIGURE 20

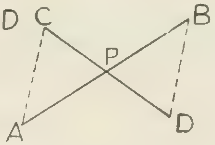

FIGURE 21

(c) In Fig. 20 ∠1 = ∠3 and ∠2 = ∠4; prove the triangles ABC and ACD are congruent.

(d) In Fig. 21 P is the middle point of AB and of CD. Prove that the triangles APC and BPD are congruent. Exercises like these are found near the beginning of many modern texts.

(3) **Exercises relating to industry.**

It is the duty of every teacher to connect the content of geometry with practical life as much as possible. In all the texts there are geometric exercises relating to carpentry, architecture, surveying, and many other industries or professions. No one author can know enough about all branches of industry to make these exercises as comprehensive as they should be. The teacher should endeavor to supplement this part of the course with other applications within his own experience and should consult tradesmen, mechanics, etc., in order to discover new applications.

(4) **Exercises involving computation.**

In the discussion of the second objective set forth in Chapter XII (§93) it was brought out that students have sometimes had the "ideal" nature of geometric elements so emphasized that they have not realized the possibility of connecting the statements of the theorems with numerical measures. This state of affairs can be prevented by the frequent use of exercises involving the application of the geometric facts in numerical computation. For examples see the discussion of the second objective in §93. The use of exercises in computation will broaden the pupil's conception of the meaning and use of the facts of geometry and it will at the same time help the pupil to retrieve his loss of skill in the processes of arithmetic and algebra. Those processes rapidly become obsolescent as far as the pupil is concerned if not reviewed and used. Thus, even at the risk of introducing algebraic and arithmetic matter apparently foreign to the subject, we should have exercises wherever possible that will encourage a review of those processes of arithmetic and algebra in which it is worth while to develop and retain skill.

References

Schultze, The Teaching of Mathematics, Chap. VII.
Smith, The Teaching of Geometry, Chap. XIII.
Young, The Teaching of Mathematics, Chap. XIII.

CHAPTER XV

CONDUCTING THE RECITATION

111. The Purpose of this Chapter. It is not the purpose of this chapter to give detailed directions concerning what should be done first, second, third, etc., during a recitation period, nor how much time should be given to each division of the recitation. Giving such advice is too much like selling a "patent medicine". A fixed recitation plan can not be used by all teachers with any more assurance than can a patent medicine be expected to cure all the diseases listed on the label. To give advice in such a manner is equivalent to suggesting limitations on a teacher's independence and originality. What is done at any particular time during a recitation depends on the nature of the subject matter in that day's lesson, the nature of the class, and the methods of the teacher. The recitation program may not always be chronologically the same. A wise teacher soon learns to do the right thing at the right time. It is sometimes necessary to break away from an outlined recitation plan for the sake of making a point—to step aside from the beaten path and meet a situation unexpectedly arising which, because of the manifest interest of the class, furnishes rare opportunity to "drive home" the principles of the subject.

It is the purpose of this chapter to deal in a general way with methods of presenting the subject matter of geometry in such a manner as to accomplish the aims set forth in Chapter XII.

112. How to Teach the Proved Propositions. In a certain geometry class a girl was reading on the blackboard her written proof of a proposition when a pupil asked, "Why is Axiom 5 used as a reason for that statement?" The teacher indicated that the girl should answer. "I do not know why", she said, and then added with an air of finality, "but I know it is by Axiom 5." That girl should have been trained to expect to do something during the recitation period which would test her understanding of the theorem and the proof of it rather than merely to be able to write down the statements of the book.

There are so many better and briefer ways to measure a student's comprehension of a theorem that it seems a pity for so many teachers to cling to the timeworn method of having him write out the proof on the blackboard as given in the text. It is easy to devise questions about the theorem and its proof, the answers to which will reveal more quickly and accurately the student's insight into the proof than will the process of writing down the successive steps of a proof that he may have memorized. Illustrations of other methods may be found in the following paragraphs.

Suppose a class has just studied the theorem, "The sum of the interior angles of a triangle is equal to two right angles." Let us assume that the method of proof in the book used a figure like Fig. 22. The class should be required to write the proof in recitation with a figure like Fig 23, or with some other similar variation of the figure. A more difficult test would be to give the figure of the triangle with the auxiliary line through the vertex parallel to the base as in Fig 24. The student who can not complete the proof with simple changes of the figure has really failed to comprehend the proof as he should. The better students of the class should be able to prove the theorem by the use of both variations of the figure as suggested above. Less than half the class period will be required for them to write the proofs and all have recited instead of a few at the blackboard. The remainder of the period can be spent in encouraging thought by developing in a heuristic manner the theorems ahead, or used for the purpose of explaining parts of the lesson for the day not clearly understood. If the pupils learn to expect this kind of a recitation instead of the too frequent order to a few, "Go to the

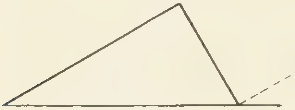

FIGURE 22

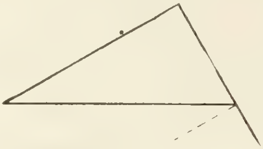

FIGURE 23

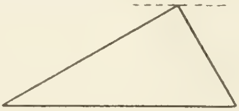

FIGURE 24

board and put on Proposition X" (or some other number), they will do something else besides memorize, without thinking about them, the statements in the book. It can not be done that way on every theorem and need not be done every day, but frequent recitations like that will not only inspire in the pupils greater effort to understand the matter in the text, but will tell a teacher more about his pupils' understanding of the proofs than anything else, except possibly the verbal replies to questions in class.

Sometimes it is easy to parallel the statements of the theorem by a concrete numerical example which may be given for the pupils to consider instead of the abstract statements of the theorem itself. For example, suppose the theorem is, "The bisectors of two supplementary adjacent angles are perpendicular to each other." For the recitation (to be written by all) the following fact may be assigned to be proved by deduction. The line OB makes an angle of 66° with the line AC. OP and OQ bisect the angles AOB and BOC. Prove that the angle POQ is a right angle. The successive steps of the proof would be: $\angle BOC = (180° - 66°) = 114°$; $\angle BOQ = 57°$; $\angle BOP = 33°$; $\angle POQ = (57° + 33°) = 90°$. The general proof has parallel statements.

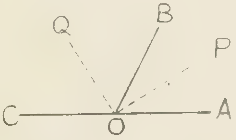

FIGURE 25

As another illustration, consider the theorem, "The perimeters of two similar polygons have the same ratio as two homologous sides." The following written exercise may be assigned as that part of the recitation which seeks to discover whether or not the proof of the theorem is understood. "Given the polygon ABCDE with the sides, AB = 4, BC = 6, CD = 10, DE = 8, EA =12; a similar polygon, A′ B′ C′ D′ E′ has the side A′ B′ = 6; prove that the ratio of the perimeter of ABCDE to the perimeter of A′ B′ C′ D′ E′ is 2/3.

A wise teacher will usually use concrete, numerical examples like those suggested in the preceding paragraphs in the introduction of the theorems to the class before assigning them. This was discussed in Chapter VI under "Inductive Method".

Let it not be understood that my purpose is in anyway to minimize the value of the model proofs of the theorems as usually given in a text or the knowledge of those proofs by the pupil. The purpose is to suggest methods of recitation that will emphasize the comprehension of the proof rather than encourage the memorization of the proof in the text. A part of the training in geometry (but not so great a part as some teachers make it) should certainly be for the purpose of enabling the pupils to set down in logical order the steps in the proofs of the fundamental theorems.

113. **Efficiency in Provoking Thought.** A proper method of measuring a teacher's efficiency in securing mental activity on the part of the pupils would be something like the following. Let us assume there are thirty students in a class and that the effective recitation period (eliminating time for classroom changing) is forty minutes. There are 1200 "student-minutes" of thinking possible if thirty pupils are engaged during the entire period in productive thought. Suppose the pupils spend on the average about thirty minutes of this time actually thinking about geometry. This makes 900 "student-minutes" of thinking. The efficiency is .75.

The question immediately arises, "How can this efficiency be increased?" It certainly does not tend to increase the number of student-minutes of thinking for half the members of the class to lounge in their seats while waiting for the other half to write out proofs of propositions on the blackboard. The seated students should be working on some original exercise while waiting. It does not increase very much the number of student-minutes of serious thought for twenty-nine students to sit and listen to one student read what he has written on the board as they attempt also to follow the reading which in many cases, because of lack of space for writing, is a difficult task in itself. Better teachers have only the figures drawn on the board, the remainder of the class being occupied with some new work during this process. Then, with all at attention, each student gives orally the proof of the theorem assigned to him, the class being kept on the alert by knowing from past experience that any one of them at any time may be called upon to "take the words out of the mouth of the speaker" in giving a reason or the next statement. The students understand that their success or failure in their responses to these "snap questions" has a decided bearing on what the teacher thinks of their ability.

Some teachers have been successful in avoiding loss of time during roll-call, etc., by having an exercise for the class written on the board before the class comes into the room. Each student is to take his seat and begin work as soon as he enters the room. This method has the double advantage of increasing efficiency and avoiding much disorder that frequently attends the assembling of a class. The exercise may be a numerical one designed to test for an appreciation of the fact that the abstract statement about geometric relation as given in the theorem applies to concrete things. It may be an easy "original theorem" or a new construction based on the theorem of the day. It may be a preliminary exercise introducing a line of inductive thought leading to the development of the proposition for the next lesson. Frequently the teacher can, by passing among the pupils as they work, make sufficient observation of the nature of each individual's work that it will not be necessary to take up the papers.

In the matter of provoking thought, no device can supplant a heuristic attitude on the part of the teacher. His bearing in the classroom should radiate questions and should inspire the pupils to do their best to answer them. The pupils should be made to feel that the business of the teacher is not so much to answer their questions as to lead them to find the answers. Suppose, for example, that a student attempts to prove that "the sum of the interior angles of a triangle is equal to two right angles". He has a mental picture of a figure like Fig. 26 but does not know how to draw the auxiliary line through B. He has "forgotten" that it should be parallel to AC. The following conversation might occur.

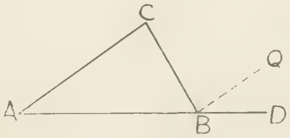

FIGURE 26

Pupil: I do not know how to draw the line, BQ.

Teacher: (Instead of telling him it should be parallel to AC) For what purpose do you draw the line?

P. In order to get equal angles.

T. To get what angles equal?

P. To get the angles that BQ makes in the angle CBD equal to the angles A and C.

T. Are the angles C and QBC situated with respect to each other in any manner that reminds you of a theorem about equal angles?

P. They look like alternate-interior angles.

T. Under what condition are such angles equal?

P. A transversal must cut two parallel lines.

T. How then would draw BQ?

P. Parallel to AC.

Every teacher should cultivate the heuristic attitude and learn how to ask the right kind of questions.

114. Measuring Progress. A teacher's estimate of a pupil's ability in geometry or any other course in mathematics should not be confined to the average of a series of numerical grades across the page of class book. It should not depend too much on the grade made in a final examination, although that certainly must have weight. There will necessarily be many grades recorded. Some means of obtaining expressions from the students of the class, as a basis for grading, have been suggested on preceding pages; for example, solving a numerical problem, proving an assigned proposition a new way, or proving an original exercise. These grades furnish a good index of a pupil's ability; but a skillful teacher can discover very much more by the intelligent nature of the replies of the student to his questions day after day. What teacher of experience has not seen with a thrill the growth of initiative, independence of thought, reasoning ability, increased confidence in his own powers, ability to make application of the subject matter, and (with it all) a constantly growing interest in the subject, by pupil after pupil in his class! If a teacher has conscientiously kept close to the thoughts of his class in geometry, he needs no average of numerical grades nor final examination to determine their standing (with some exceptions) unless it be merely to confirm his estimates and assure him that his grades are fair. He should certainly be able to discover more about a pupil in four or five months of daily conversation, consisting of questions and answers, than can be discovered in the two or three hours alloted to an examination. This continued appraisal of a student is part of the teacher's work in every recitation.

Let this additional word be said about one phase of an examination, or written test, or "quiz." If the students are to be tested on their ability to prove original exercises, the test should not include any of the exercises studied in the text. We should choose instead, some exercises which the class has never seen

If a limited time is allowed for the test, the exercises must not be very difficult, nor should the construction of the necessary figures require a great deal of time. We do not mean to imply that the pupils should not be tested on their ability to prove the fundamental theorems and various minor theorems or exercises having important practical applications, but this is too often a test of memory rather than a test of initiative in the use of geometric facts to establish other facts.

115. Teaching Scientific English. So much modern reading is for thrills instead of thought that a student needs to be taught how to read and express statements that require thought. A text in mathematics can not be scanned at a glance like a typical "continued on page 148" magazine story where the reader can easily trace the run of the plot and scan over a great deal of "padding" put in by the writer. There are no superfluous adjectives in scientific English. A reader must go slow and think. The pupils can learn to appreciate the beauty and simplicity of this type of writing by learning to write and talk that way. They should learn to "say what they mean."

One way to encourage care and accuracy in making statements is to have a considerable amount of work done at the board at the dictation of individual students, using a plan of procedure much like the following. Let us assume that the teacher is asking the pupil to suggest the necessary figure in order to develop some theorem about a circle. A circle is drawn, a radius is drawn, and the student says, "Now draw a tangent perpendicular to the radius". The teacher does so, at some interior point. "No", exclaims the student "I mean at the end of the radius." The teacher then draws the perpendicular through the center of the circle and the pupil is taught he should have specified at the beginning that the perpendicular be drawn at the point of intersection of the radius with the circle. If managed so as not to emphasize too much the ludicrous situations that will naturally arise and cause a loss of teaching value, this method of doing something not intended by the pupil but which may fulfill his instructions can be made of great value in producing definite and accurate statements. There are other schemes which accomplish the same purpose. Some teachers have done a similar thing by drawing a figure fulfilling the definition given by a pupil but which is not the figure intended by the pupil. Indeed, it is sometimes possible to follow the instructions given by the definitions in the text and draw figures that would surprise the author.

It is not my purpose to emphasize any particular scheme, but to make clear that it is the duty of the teacher of geometry to insist on accurate statements of definitions and theorems (not necessarily memorized from the book) and thereby assist in the teaching of English. Indeed, the teacher of every subject in the curriculum should do his bit in the teaching of English.

References

Smith, The Teaching of Geometry, Chap. X.

CHAPTER XVI

A SUMMARY OF STANDARDS AND OBJECTIVES FOR TEACHERS

116. What a Good Teacher Should Know. As the title indicates, this chapter is a summary of the preceding chapters. The ten outstanding statements made here are as from an older teacher to a young one. They are not meant as commandments (though there are exactly ten) but as exhortations (such as Paul gave to Timothy). Teachers of experience are asked to be lenient in their judgment of the attitude taken here.

I. **Know very much more about the subject than you can teach your pupils and continue learning about it.**

This needs little discussion here. It is the keynote of Part One of this bulletin. Two illustrations will be added here.

A young man who was ranked very high in mathematical ability had almost completed the work for a Master's degree, majoring in mathematics, when he secured a position as teacher of mathematics in high school. His general attitude, manner of speech, and reactions toward educational methods were such that it seemed as if he might be a failure as a teacher of boys and girls. He had a minimum preparation along the lines of psychology and education. At the end of the first year his superintendent was asked for a confidential statement concerning the young man's work as a teacher. He wrote as follows:

> His only weakness in high school is the problem of discipline. He gets so interested in the development of the subject matter that he doe not notice slight disturbances at all. It is a fact that he can get more mathematics across to the students in spite of trival disturbances from time to time than I have ever been able to do and I have been regarded as a strict disciplinarian.

A few years ago thousands of students in the high schools of the North Central Association of Colleges and Secondary Schools were asked to name that quality, or item of professional equipment, which they most desired in their teachers. That quality which was named by more pupils than any other was "knowledge of the subject".

II. **Know the old and new ideas of arrangement of subject matter and methods of teaching.**

A study of the changes that have taken place in the methods of presenting mathematics ought to convince a teacher that he can not afford to be dogmatical in his attitude toward methods of teaching the subject. It ought also to inspire in him an ambition to do some independent thinking, make some investigations, conduct some experiments, and thereby possibly make some contribution himself to the cause. Methods of teaching mathematics are no more irrevocable than are the postulates of geometry.

III. **Know the history of the development of each subject you teach.**

This will help your pedagogy if there is any truth in the theory that a child lives over again the mental life of the race. It will give you a source from which to draw material to enliven your teaching. It will certainly broaden your conception of the subject.

117. What a Good Teacher Should Do—for Himself.

I. **Maintain membership in at least one organization for mathematics teachers, subscribing for and reading at least one journal devoted to mathematical interests.**

A body of water that is not replenished from a fresh source becomes stagnant. It is not good for one's health to drink from it. A teacher owes it to himself to keep abreast of the times to keep from "drying up". When leaves fail to receive further sustenance from the tree they dry up and blow away. The teacher who neglects to keep in touch with the progress his best colleagues are making in the improvement of methods of teaching will also find his ability and mental capacity shriveling. Unfortunately for the great cause of education he is not like the leaf. He does not blow away.

The only national organization that exists primarily for teachers of high school mathematics is the National Council of Teachers of Mathematics. The membership fee in this organization is two dollars per year and includes a subscription to the "Mathematics Teacher", published under the suspices of the National Council in the interest of secondary mathematics. John R. Clark, Lincoln School, Columbia University, New York City, is the editor and publisher of the "Mathematics Teacher".

The Central Association of Science and Mathematics Teachers has for years had "School Science and Mathematics" as its official journal. The annual membership fee of three dollars includes a subscription to the journal which is published by Smith and Turton, 2055 East 72d Place, Chicago. The annual subscription price of "School Science and Mathematics" apart from membership in the Central Association is $2.50. It covers the field of high school science as well as mathematics.

For high school teachers of ability and ambition there is the Mathematical Association of America, membership in which is by election on the recommendation of two members. This Association was organized in the interest of collegiate mathematics and a great deal of emphasis is placed by the Association on methods of teaching. The articles in its official journal, the "American Mathematical Monthly", which deal with pure mathematics can be read by any one who has had mathematics through calculus. Very few of the articles require even that preparation. The Association requires an initiation fee of two dollars and annual dues of four dollars, which include a subscription to the "Monthly". The price of the "Monthly" to non-members is five dollars per year. The Treasurer of the Association is W. D. Cairns, Oberlin, Ohio.

II. **Do not copy slavishly the methods of others. Develop your own.**

This does not preclude the possibility of trying out, with probable variations, the methods of others that appeal to you. By all means do not copy the exact methods of your college and university teachers without careful consideration. They would not use the same methods with high school pupils.

III. **Make yourself skillful in as many incidental operations as possible and teach without your text as much as possible for it inspires confidence.**

The story related in the opening chapter shows the attitude of many minds toward mathematics. The teacher who can add, subtract, multiply, and divide the ordinary numbers of arithmetic with considerable speed and accuracy will inspire confidence in his mathematical ability". They will regard him as a good "mathematician". It is hoped that he will teach them a better conception of the term "mathematician" but he may as well capitalize his skill in computation. A class that has confidence in the skill and ability of its teacher will give better attention. If a teacher knows, say, methods of factoring a trinomial like $6x^2-13x+6$ other than those mentioned in the book, the suggestion of one of them to some of the best members of the class who have the capacity for more than the class is doing as a whole will not only be a service to the pupils but will redound to the honor of the teacher "who knows more than is in the book". There are many opportunities like that coming up all the time. I know a teacher who can draw a nearly perfect circle with one sweep of the crayon (a little practice using the elbow as center and the forearm as radius, allowing the arm to turn at the shoulder, will soon get it) who never fails to get a gasp of astonishment from the new class when they first observe it and then they give attention to see what will happen next.

My first superintendent said to us one day in teachers' meeting: "I notice two attitudes of teachers toward textbooks as betrayed by their classes. When I visit one type of teacher she promptly hands me the text she is using at the time and several pupils at once offer their texts to her, seemingly knowing that she must have one. There is another kind who has no text in hand when I enter and the pupils offer me theirs. They know from experience that she needs none." What a lesson to be learned from that!

118. What a Good Teacher Should Do—for His Pupils.

I. Make sure that fundamental processes are clearly understood and not applied as by "rule of thumb".

II. Develop a certain degree of mechanical efficiency in the processes used most. It is secured only through the solution of many exercises.

I and II have been fully discussed in §83.

III. Relate the subject matter in every way possible to practical applications.

This needs no discussion.

IV. Aim to develop independent thinkers with initiative and resourcefulness, whose ideals are precision of thought, clearness and accuracy of statement, and efficiency in deeds.

APPENDIX

I. BIBLIOGRAPHY

In the following list of the references mentioned in this bulletin those printed in black type are the first in each classification for a teacher to buy (according to my opinion), either for himself or the high school library. Those marked with asterisks would make a well balanced initial collection for a high school library.

A. History of Mathematics in General:

D. E. Smith, History of Mathematics, 2 vols. Ginn.

*Vol. I, $4.00.

A general survey of elementary mathematical science arranged chronologically.

***Vol. II,** $4.40.

A topical survey, treating of the history of special branches, such as numbers, computation, algebra, geometry.

***Florian Cajori, History of Mathematics,** Macmillan, $4.00.

*Cajori, History of Elementary Mathematics, Macmillan, $2.25.

*G. A. Miller, Historical Introduction to Mathematical Literature, Macmillan.

W. W. R. Ball, History of Mathematics, Macmillan, $4.00.

C. Fink, A Brief History of Mathematics, Translated by Beman and Smith, Open Court Pub. Co., Chicago, $1.50.

B. History of Special Topics in Mathematics:

***T. L. Heath, History of Greek Mathematics,** 2 vols. Clarendon Press, $16.70.

The standard authority on the subject.

*Smith, Number Stories of Long Ago, Ginn, 60 cents.

A reading book for elementary pupils.

Smith-Karpinski, The Hindu-Arabic Numerals, Ginn, $1.60.

Smith-Mikami, History of Japanese Mathematics, Open Court, $2.00.

*James Gow, A Short History of Greek Mathematics, Cambridge University Press.

L. L. Conant, Number Concept, Macmillan, $2.40.

T. Eric Peet, The Rhind Mathematical Papyrus, The University Press of Liverpool, (expensive).

Tylor, Primitive Culture, Vol. I, John Murray, London.

Hobson (See under D, Geometry.)

Rupert(See under D, Geometry.)
Fine (See under C, Algebra.)

C. **Algebra:**

***Henry B. Fine, The Number System of Algebra, Heath.**

Part II is historical. Except for the historical part, Fine's "College Algebra", Ginn & Co., may be substituted for this.

R. Dedekind, Essays on Number, Translated by Beman, Open Court, 75 cents.

H. P. Manning, Irrational Numbers, Wiley.

D. **Geometry:**

***Heath, The Thirteen Books of Euclid's Elements,** Cambridge University Press, about $15.00.

***W. W. Rupert, Famous Geometrical Theorems and Problems,** Parts I-IV, Heath (inexpensive).

***Somerville, Non-Euclidean Geometry.**

H. P. Manning, Non-Euclidean Geometry, Ginn, $1.36.

E. W. Hobson, Squaring the Circle, Cambridge University Press.

See also J. W. and J. W. A. Young, under F, General References.

E. **Teaching of Mathematics:**

***A. Schultze, The Teaching of Mathematics in Secondary Schools,** Macmillan, $1.80.

*J. W. A. Young, The Teaching of Mathematics, Revised Edition, 1924, Longmans, $2.20.

*H. C. Barber, Teaching Junior High School Mathematics, Houghton Mifflin, $1.20.

*T. S. Row, Geometric Exercises in Paper Folding, Open Court, $1.00.

*J. R. Clark, Mathematics in the Junior High School, The Gazette Press, Yonkers, N. Y.

This is a preliminary edition and may not be obtainable until the final edition is out. Every teacher of junior high school mathematics should have it as soon as obtainable.

*Paul Ligda, The Teaching of Elementary Algebra, Houghton Mifflin.

N. J. Lennes, The Teaching of Arithmetic, Macmillan, $2.00.

J. R. Overman, Principles and Methods of Teaching Arithmetic, Lyons and Carnahan, $1.50.

S. C. Sumner, Supervised Study in Mathematics and Science, Macmillan, $1.40.

T. P. Nunn, The Teaching of Algebra, Including Trigonometry, Longmans, Vol. I, $2.25.

A. W. Stamper, A History of the Teaching of Elementary Geometry, Published by Teachers College, Columbia University.

This may not be obtainable.

Smith, The Teaching of Geometry, Ginn.

This may be out of print.

F. **General References:**

*J. W. Young, **Lectures on Fundamental Concepts of Algebra and Geometry,** Macmillan, $2.00.

This gives a clear conception of the foundations of algebra and geometry with illustrations of non-Euclidean geometry. There is also a valuable chapter at the end on "The growth of Algebraic Symbolism" by U. G. Mitchell. Every forward-looking teacher of mathematics should read this book.

*J. W. A. Young, **Monographs on Topics of Modern Mathematics Relevant to the Elementary Field,** Second Edition, 1924, Longmans, $4.00.

Every high school teacher of mathematics should have this book. It considers such questions as the foundations and fundamental concepts of algebra and geometry, the general nature of the algebraic equation, non-Euclidean geometry, and the history and transcedence of π.

*Reorganization of Mathematics in Secondary Education, being the report of the National Committee. For information, see §72.

G. **Recreations in Mathematics:**

*Ball, Mathematical Recreations, Tenth Edition, Macmillan, $3.50.

*Manning, The Fourth Dimension Simply Explained, Munn & Co., N. Y.

"A Square" (P. Abbott), Flatland, Little, Brown & Co., $1.25.

Treats of the Fourth Dimension.

S. I. Jones, Mathematical Wrinkles, S. I. Jones, Gunter, Texas, $2.10.

H. E. Licks, Recreations in Mathematics, Van Nostrand, $1.50.

H. E. Dundenny, Amusements in Mathematics, Thomas Nelson & Sons, $1.75.

H. **Mathematical Journals:**

See references in §117.

II. LIST OF PROPOSITIONS IN PLANE GEOMETRY

The following list is reprinted from Chapter VI of the Report of the National Committee. The theorems marked with asterisks are theorems so marked by the College Entrance Examination Board in its recommended list of ninety-one theorems.

The Board announces that the "starred" theorems are the only ones for which proof will be asked in future examinations and adds:

> The starred propositions have been chosen partly with regard to their intrinsic inportance, but partly also with regard to their availability for examination purposes. A theorem may be of the highest importance, and yet may have a proof which, for one reason or another, is of such a nature that little information as to the candidate's ability is obtained by asking him to reproduce it. On this account, some of the most important propositions in the course . . . . . are omitted from the list.

The College Entrance Examination Board has two starred theorems not mentioned in the list below; namely, "Through any three given points not lying in the same straight line one circle, and only one can be drawn" and "a circle may be circumscribed about any regular polygon".

The stars on the numbers of the National Committee's list here include all theorems listed under the numbers except where noted.

General basis of the selection of material.[61]—The subcommittee appointed to prepare a list of basal propositions made a careful study of a number of widely used textbooks on geometry. The bases of selection of the propositions were two: (1) The extent to which the propositions and corollaries were used in subsequent proofs of important propositions and exercises: (2) the value of the propositions in completing important pieces of theory. Although the list of theorems and problems is substantially the same in nearly all textbooks in general use in this country, the wording, the sequence, and the methods of proof vary to such an extent as to render difficult a definite statement as to the number of times a proposition is used in the several books examined. A tentative table showed, however, less variation than might have been anticipated.

* * * * * * * * * *

PLANE GEOMETRY

I. **Assumptions and theorems for informal treatment.**—This list contains propositions which may be assumed without proof (postulates), and theorems which it is permissible to treat informally. Some of these propositions will appear as definitions in certain methods of treatment. Moreover, teachers should feel free to require formal proofs in certain cases, if they desire to do so. The precise wording given is not essential, nor is the order in which the propositions are here listed. The list should be taken as representative of the type of propisitions which may be assumed, or treated informally, rather than as exhaustive.

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[61] Everything from here on is quoted from the Report of the National Committee, pp. 55-60.

1. Through two distinct points it is possible to draw one straight line, and only one.

2. A line segment may be produced to any desired length.

3. The shortest path between two points is the line segment joining them.

4. One and only one perpendicular can be drawn through a given point to a given straight line.

5. The shortest distance from a point to a line is the perpendicular distance from the point to the line.

6. From a given center and with a given radius one and only one circle can be described in a plane.

7. A straight line intersects a circle in at most two points.

8. Any figure may be moved from one place to another without changing its shape or size.

9. All right angles are equal.

10. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.

11. Equal angles have equal complements and equal supplements.

12. Vertical angles are equal.

13. Two lines perpendicular to the same line are parallel.

14*. Through a given point not on a given straight line one straight line, and only one, can be drawn parallel to the given line.

15. Two lines parallel to the same line are parallel to each other.

16. The area of a rectangle is equal to its base times its altitude.

II. **Fundamental theorems and constructions.**—It is recommended that the theorems and constructions (other than originals) to be proved on college entrance examinations be chosen from the following list. Originals and other exercises should be capable of solution by direct reference to one or more of these propositions and constructions. It should be obvious that any course in geometry that is capable of giving adequate training must include considerable additional material. The order here given is not intended to signify anything as to the order of presentation. It should be clearly understood that certain of the statements contain two or more theorems, and that the precise wording is not essential. The committee favors entire freedom in statement and sequence.

A. THEOREMS.

1*. Two triangles are congruent if[1] *(a)* two sides and the included

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[1] Teachers should feel free to separate this theorem into three distinct theorems and to use other phraseology for any such proposition. For example, in 1, "Two triangles are equal if" * * * "a triangle is determined by * * *,' etc. Similarly in 2, the statement might read: "Two right tri-

angles are congruent if, beside the right angles, any two parts (not both angles) in the one are equal to corresponding parts of the other."
angle of one are equal, respectively, to two sides and the included angle of the other; *(b)* two angles and a side of one are equal, respectively, to two angles and the corresponding side of the other; *(c)* the three sides of one are equal, respectively, to the three sides of the other.

2*. Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other.

3*. If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely.[2]

[2] It should be understood that the converse of a theorem need not be treated in connection with the theorem itself, it being sometimes better to treat it later. Furthermore a converse may occasionally be accepted as true in an elementary course, if the necessity for proof is made clear. The proof may then be given later.

4*. The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line segment joining them.

5*. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.

6*. When a transversal cuts two parallel lines, the alternate interior angles are equal; and conversely. (Converse is not starred.)

7*. The sum of the angles of a triangle is two right angles.

8. A parallelogram is divided into congruent triangles by either diagonal.

9* Any (convex) quadrilateral is a parallelogram *(a)* if the opposite sides are equal; *(b)* if two sides are equal and parallel.

10*. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.

11. *(a)** The area of a parallelogram is equal to the base times the altitude.

(b) The area of a triangle is equal to one-half the base times the altitude.

*(c)** The area of a trapezoid is equal to half the sum of its bases times its altitude.

*(d)** The area of a regular polygon is equal to half the product of its apothem and perimeter.

12. *(a)* If a straight line is drawn through two sides of a triangle parallel to the third side, it divides these sides proportionally.

*(b)** If a line divides two sides of a triangle proportionally, it is parallel to the third side. (Proofs for commensurable cases only.)

(c) The segments cut off on two transversals by a series of parallels are proportional.

13*. Two triangles are similar if *(a)* they have two angles of one equal, respectively, to two angles of the other; *(b)* they have an angle of one equal to an angle of the other and the including sides are proportional; *(c)* their sides are respectively proportional.

14*. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.

15. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

16* Polygons are similar, if they can be decomposed into triangles which are similar and similarly placed; and conversely. (Only the converse is starred.)

17. The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent sides.

18*. The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.

19. In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles each similar to the given triangle.

20*. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

21. In the same circle, or in equal circles, if two arcs are equal, their central angles are eqúal; and conversely.

22. In any circle angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only.)

23*. In the same circle or in equal circles, if two chords are equal their corresponding arcs are equal; and conversely.

24. *(a)** A diameter perpendicular to a chord bisects the chord and the arcs of the chord. *(b)*. A diameter which bisects a chord (that is not a diameter) is perpendicular to it.

25. The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.

26. In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.

27*. An angle inscribed in a circle is equal to half the central angle having the same arc.

28. Angles inscribed in the same segment are equal.

29. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon and tangents at the points of division form a regular circumscribed polygon.

30. The circumference of a circle is equal to $2\pi r$. (Informed proof only.)

31.[3] The area of a circle is equal to πr^2. (Informal proof only.)

The treatment of the mensuration of the circle should be based upon related theorems concerning regular polygons, but it should be informal as to the limiting processes involved. The aim should be an understanding of the concepts involved, so far as the capacity of the pupil permits.

[3] The total number of theorems given in this list when separated, as will probably be formed advantageous in teaching, including the converses indicated, is 52.

B. CONSTRUCTIONS

1. Bisect a line segment and draw the perpendicular bisector.
2. Bisect an angle.
3. Construct a perpendicular to a given line through a given point.
4. Construct an angle equal to a given angle.
5. Through a given point draw a straight line parallel to a given straight line.
6. Construct a triangle, given *(a)* the three sides; *(b)* two sides and the included angle; *(c)* two angles and the included side.
7. Divide a line segment into parts proportional to given segments.
8. Given an arc of a circle, find its center.
9. Circumscribe a circle about a triangle.
10. Inscribe a circle in a triangle.
11. Construct a tangent to a circle through a given point.

12. Construct the fourth proportional to three given line segments.
13. Construct the mean proportional between two given line segments.
14. Construct a triangle (polygon) similar to a given triangle (polygon).
15. Construct a triangle equal to a given polygon.
16. Inscribe a square in a circle.
17. Inscribe a regular hexagon in a circle.

III. **Subsidiary list of propositions.**—The following list of propositions is intended to suggest some of the additional material referred to in the introductory paragraphs of Section II. It is not intended, however, to be exhaustive; indeed, the committee feels that teachers should be allowed considerable freedom in the selection of such additional material, theorems, corollaries, originals, exercises, etc., in the hope that opportunity will thus be afforded for constructive work in the development of courses in geometry.

1. When two lines are cut by a transversal, if the corresponding angles are equal, or if the interior angles on the same side of the transversal are supplementary, the lines are parallel.

2. When a transversal cuts two parallel lines, the corresponding angles are equal, and the interior angles on the same side of the transversal are supplementary.

3. A line perpendicular to one of two parallels is perpendicular to the other also.

4. If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.

5. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.

6. The sum of the angles of a convex polygon of n sides is $2(n-2)$ right angles.

7. In any parallelogram *(a)* the opposite sides are equal; *(b)* the opposite angles are equal; *(c)* the diagonals bisect each other.

8. Any (convex) quadrilateral is a parallelogram, if *(a)* the opposite angles are equal; *(b)* the diagonals bisect each other.

9. The medians of a triangle intersect in a point which is two-thirds of the distance from a vertex to the mid-point of the opposite side.

10. The altitudes of a triangle meet in a point.

11. The perpendicular bisectors of the sides of a triangle meet in a point.

12. The bisectors of the angles of a triangle meet in a point.

13. The tangents to a circle from an external point are equal.

14.[4] *(a)* If two sides of a triangle are unequal, the greater side has the greater angle opposite it, and conversely.

[4] Such inequality theorems as these are of importance in developing the notion of dependence or functionality in geometry. The fact that they are placed in the "Subsidiary list of propositions" should not imply that they are considered of less educational value than those in List II. They are placed hree because they are not "fundamental" in the same sense that the theorems of List II are fundamental.

(b) If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the in-

cluded angle of the second, then the third side of the first is greater than the third side of the second, and conversely.

(c) If two chords are unequal, the greater is at the less distance from the center, and conversely.

(d) The greater of two minor arcs has the greater chord, and conversely.

15. An angle inscribed in a semicircle is a right angle.

16. Parallel lines tangent to or cutting a circle intercept equal arcs on the circle.

17. An angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.

18. An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.

19. An angle formed by two secants or by two tangents to a circle is measured by half the difference between the intercepted arcs.

20. If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

21. Parallelograms or triangles of equal bases and altitudes are equal.

22. The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems.

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